In Image and Video Communications and Processing 2003, Bhaskaran Vasudev, T. Russell Hsing, Andrew G. Tescher, Touradj Ebrahimi, eds., Proceedings of SPIE, vol. **5022**, pp. 75–86, May 2003.

Estimation of multiple motions: regularization and performance evaluation

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ABSTRACT

This paper deals with the problem of estimating multiple transparent motions that can occur in computer vision applications, e.g. in the case of semi-transparencies and occlusions, and also in medical imaging when different layers of tissue move independently. Methods based on the known optical-flow equation for two motions are extended in three ways. Firstly, we include a regularization term to cope with sparse flow fields. We obtain an Euler-Lagrange system of differential equations that becomes linear due to the use of the mixed motion parameters. The system of equations is solved for the mixed-motion parameters in analogy to the case of only one motion. To extract the motion parameters, the velocity vectors are treated as complex numbers and are obtained as the roots of a complex polynomial of a degree that is equal to the number of overlaid motions. Secondly, we extend a Fourier-Transform based method proposed by Vernon such as to obtain analytic solutions for more than two motions. Thirdly, we not only solve for the overlaid motions but also separate the moving layers. Performance is demonstrated by using synthetic and real sequences.

1. INTRODUCTION

This paper addresses the problem of estimating multiple transparent motions that can occur in computer-vision applications, e.g. in case of semi-transparencies and occlusions, and also in medical x-ray projections imaging, when different layers of tissue move independently. An overview of the problem of multiple motions has been given in [1]. To our knowledge, the problem of two motions has been first solved in [2] by the use of spatio-temporal Gabor filters and fourth-order moments derived from these filters. An alternative solution that is also based on the frequency domain is given in [3], where a nonlinear system of four equations is solved to estimate the phase change and from there two transparent motions. In general, frequency-based methods suffer from requiring large local windows. Others have introduced the useful and intuitive notion of 'layers'.[4] As an important extension of the methods mentioned above, we have provided analytic solutions for up to four transparent motions.[5] Our approach also delivers numerical solutions for more than four motions. Here we first extend the approach to include regularization. Vernon[3] proposed a different framework for multiple transparent motions based on the phase of the Fourier components. This solution, however, is limited to only two motions. We solve the equations introduced by Vernon for the more general case of an arbitrary number of motions. We also identify some basic problems of Vernon's approach to the separation of transparent motion layers.

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2. THE THEORY OF MULTIPLE TRANSPARENT MOTIONS

2.1. Spatial model for two transparent motions

Suppose that an image sequence $f(x, y, t)$ is the overlaid superposition of two other sequences g_1, g_2 moving with constant velocities $\mathbf{u} = (u_x, u_y)^T$ and $\mathbf{v} = (v_x, v_y)^T$. The resulting f is then described as

$$f(x, y, t) = g_1(x - u_x t, y - u_y t) + g_2(x - v_x t, y - v_y t). \tag{1}$$

The task is to determine $\mathbf{v}$ and $\mathbf{u}$ given f. To do so, we use the optical-flow equation introduced by Shizawa and Mase[2]:

$$\alpha(\mathbf{u})\alpha(\mathbf{v})f = 0. \tag{2}$$

Note that the equation involves the concatenated directional derivatives along $\mathbf{u}$ and $\mathbf{v}$. After expanding the above equation we obtain the following expression:

$$\alpha(\mathbf{u})\alpha(\mathbf{v})f = f_{xx}u_x v_x + f_{yy}u_y v_y + f_{xy}(u_x v_y + u_y v_x) + f_{xt}(u_x + v_x) + f_{yt}(u_y + v_y) + f_{tt} = 0. \tag{3}$$

As in [5] we use the following notation:

$$\begin{aligned} c_{xx} &= u_x v_x & c_{yy} &= u_y v_y \\ c_{xy} &= u_x v_y + u_y v_x & c_{xt} &= u_x + v_x \\ c_{yt} &= u_y + v_y & c_{tt} &= 1. \end{aligned} \tag{4}$$

With the c's denoting the mixed-motion parameters. Eq. (2) then becomes:

$$\alpha(\mathbf{u})\alpha(\mathbf{v})f = f_{xx}c_{xx} + f_{yy}c_{yy} + f_{xy}c_{xy} + f_{xt}c_{xt} + f_{yt}c_{yt} + f_{tt}c_{tt} = 0. \tag{5}$$

As we shall see, this notation leads to a linear formulation of the problem.

The mixed-motion parameters can be separated by using the novel method described by Mota et.al.[5] The velocity vectors are thereby treated as complex numbers:

$$\mathbf{u} = u_x + ju_y, \ \mathbf{v} = v_x + jv_y. \tag{6}$$

These complex numbers are related to the mixed motion parameters c by the following equations:

$$\begin{aligned} \mathbf{uv} &= A_0 = c_{xx} - c_{yy} + jc_{xy} \\ \mathbf{u} + \mathbf{v} &= A_1 = c_{xt} + jc_{yt}. \end{aligned} \tag{7}$$

Note that A_0 and A_1 are homogeneous symmetric functions in $\mathbf{u}$ and $\mathbf{v}$ and, by Vieta's theorem, the coefficients of the complex polynomial

$$Q(z) = (z - \mathbf{u})(z - \mathbf{v}) = z^2 - A_1 z + A_0 \tag{8}$$

that has the complex roots $\mathbf{u}$ and $\mathbf{v}$. These roots can be obtained analytically (even for the case of up to four motions[5]). Thus the main steps in solving for multiple transparent motions are to (i) solve the linear system for the c's (ii) find the roots of the complex polynomial $Q(z)$, and (iii) take the real parts of the z's as x- and the imaginary parts as y-components of the motion vectors.

2.2. Spatial model for multiple transparent motions

In this section we analyze the case of more than two transparent motions. A number N of transparent motions are described by the following equation:

$$f(\mathbf{x},t) = g_1(\mathbf{x} - t\mathbf{v}_1) + g_2(\mathbf{x} - t\mathbf{v}_2) + \cdots + g_N(\mathbf{x} - t\mathbf{v}_N). \tag{9}$$

In this case, the optical-flow equation is given by

$$\alpha(\mathbf{v}_1)\cdots\alpha(\mathbf{v}_N)f = \sum_I c_I f_I = 0, \tag{10}$$

with the notation $I = I_1, \ldots, I_M$, $M = (N+1)(N+2)/2$. I_i are ordered sequences with elements (x, y, t). For example in Eq. (4) $I_1 = xx$, $I_4 = xt$, and $I_6 = tt$. As for the case of two motions, multiple motions can be separated by solving for the roots of an N-degree polynomial.[5]

3. REGULARIZATION FOR MULTIPLE MOTIONS

3.1. Regularization for two transparent motions

The regularization procedure is defined in analogy to the case of only one motion.[6] As with one motion we have only one equation for the motion parameters, but now the number of unknowns is five. We therefore need four more equations. We employ the calculus of variation and define, in analogy to the method used in by Horn and Schunck,[6] the following regularization term:

$$\begin{aligned} N = {} & (\partial_x c_{xx})^2 + (\partial_y c_{xx})^2 \\ & + (\partial_x c_{yy})^2 + (\partial_y c_{yy})^2 + (\partial_x c_{xy})^2 + (\partial_y c_{xy})^2 + (\partial_x c_{xt})^2 + (\partial_y c_{xt})^2 + (\partial_x c_{yt})^2 + (\partial_y c_{yt})^2. \end{aligned} \tag{11}$$

We thus obtain the parameters c as the values that minimize the above term together with the squared optical-flow term (2), i.e.:

$$\iint \big(\alpha(\mathbf{u})\alpha(\mathbf{v})f\big)^2 + \lambda^2 N \, \mathbf{d}\Omega, \tag{12}$$

λ is the regularization parameter and Ω the whole image plane over which we integrate. Note that, at this stage, we work on finding the c's and not the motion-vector components. This has the great advantage that we obtain an Euler-Lagrange system of differential equations that is linear. As we shall see, this would not be the case, when working directly on the motion vectors themselves. Note that if the velocities $\mathbf{u}$ and $\mathbf{v}$ are smooth, the parameters c will also be smooth. The five Euler-Lagrange equations that we obtain are the following:

$$\begin{aligned} \lambda^2 \triangle c_{xx} &= f_{xx}^2 c_{xx} + f_{xx}f_{yy}c_{yy} + f_{xx}f_{xy}c_{xy} + f_{xx}f_{xt}c_{xt} + f_{xx}f_{yt}c_{yt} + f_{xx}f_{tt} \\ \lambda^2 \triangle c_{yy} &= f_{yy}f_{xx}c_{xx} + f_{yy}^2 c_{yy} + f_{yy}f_{xy}c_{xy} + f_{yy}f_{xt}c_{xt} + f_{yy}f_{yt}c_{yt} + f_{yy}f_{tt} \\ \lambda^2 \triangle c_{xy} &= f_{xy}f_{xx}c_{xx} + f_{xy}f_{yy}c_{yy} + f_{xy}^2 c_{xy} + f_{xy}f_{xt}c_{xt} + f_{xy}f_{yt}c_{yt} + f_{xy}f_{tt} \\ \lambda^2 \triangle c_{xt} &= f_{xt}f_{xx}c_{xx} + f_{xt}f_{yy}c_{yy} + f_{xt}f_{xy}c_{xy} + f_{xt}^2 c_{xt} + f_{xt}f_{yt}c_{yt} + f_{xt}f_{tt} \\ \lambda^2 \triangle c_{yt} &= f_{yt}f_{xx}c_{xx} + f_{yt}f_{yy}c_{yy} + f_{yt}f_{xy}c_{xy} + f_{yt}f_{xt}c_{xt} + f_{yt}^2 c_{yt} + f_{yt}f_{tt}. \end{aligned} \tag{13}$$

3.1.1. Linear and nonlinear formulations of the problem

As noted above, the system (13) is linear in the c's. Let us consider the following example to illustrate that this would not be the case with the motion parameters themselves. To obtain the Euler-Lagrange differential equations, one must differentiate the functional to be minimized with respect to the unknown variables. If we differentiate only the expression of the squared optical-flow equation with respect to v_x, we obtain

$$\frac{\partial}{\partial v_x}\Big(\alpha(\mathbf{u})\alpha(\mathbf{v})f\Big)^2 = 2\Big(f_{xx}u_xv_x + f_{yy}u_yv_y + f_{xy}(u_xv_y + u_yv_x) + f_{xt}(u_x + v_x) + f_{yt}(u_y + v_y) + f_{tt}\Big)\Big(f_{xx}u_x + f_{xy}u_y + f_{xt}\Big), \quad (14)$$

i.e., we obtain an equation that is nonlinear in v_x, v_y, u_x and u_y. Therefore we note that it is indeed the introduction of the mixed motion parameters c that leads to a linear formulation of the problem.

3.1.2. Update rule

In analogy to the case of one motion we obtain the following update rules for the system (13):

$$\begin{aligned} c_{xx}^{l+1} &= \hat{c}_{xx}^{l} - f_{xx}\frac{P}{D} \\ c_{yy}^{l+1} &= \hat{c}_{yy}^{l} - f_{yy}\frac{P}{D} \\ c_{xy}^{l+1} &= \hat{c}_{xy}^{l} - f_{xy}\frac{P}{D} \\ c_{xt}^{l+1} &= \hat{c}_{xt}^{l} - f_{xt}\frac{P}{D} \\ c_{yt}^{l+1} &= \hat{c}_{yt}^{l} - f_{yt}\frac{P}{D}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} P &= f_{xx}\hat{c}_{xx}^{l} + f_{yy}\hat{c}_{yy}^{l} + f_{xy}\hat{c}_{xy}^{l} + f_{xt}\hat{c}_{xt}^{l} + f_{yt}\hat{c}_{yt}^{l} + f_{tt} \\ D &= \lambda^2 + f_{xx}^2 + f_{yy}^2 + f_{xy}^2 + f_{xt}^2 + f_{yt}^2. \end{aligned} \quad (16)$$

The iteration will deliver the values of the mixed motion parameters c. From these parameters we still need to extract the velocity vectors $\mathbf{u}$ and $\mathbf{v}$ as described in Section 2.1.

3.1.3. Boundary treatment

We have chosen to extend the image by copying the boundary pixels into an extended margin of size one. Such, the first-order derivatives will be zero outside the boundary of the original image thus minimizing boundary effects.

3.2. Regularization of multiple motions

In case of more than two motions, the functional to be minimized becomes:

$$\iint \Big(\sum_{I} c_I f\Big)^2 + \lambda^2 \sum_{I\setminus I_M}\Big((\partial_x c_I)^2 + (\partial_y c_I)^2\Big)\,\mathbf{d}\Omega \quad (17)$$

and the Euler-Lagrange differential equations are

$$\Big(\sum_{I} c_I f_I\Big) f_{I_i} = \lambda^2 \triangle c_{I_i}, \qquad i = 1, \cdots, M-1. \quad (18)$$

Discretization with $\triangle c_{I_i} = \hat{c}_{I_i} - c_{I_i}$ leads to

$$\Big(\sum_{I\setminus I_M} c_I f_I\Big) f_{I_i} + \lambda^2 c_{I_i} = \lambda^2 \hat{c}_{I_i} - f_{I_i} f_{I_M}, \qquad i = 1, \cdots, M-1. \quad (19)$$

It is now straightforward to show that Equation (19) leads to the following update rule:

$$c_{I_i}^{l+1} = \hat{c}_{I_i}^{l} - f_{I_i}\frac{P}{D}, \qquad i = 1, \cdots, M-1, \tag{20}$$

where

$$P = \sum_{I} \hat{c}_I^l f_I \qquad \text{and} \qquad D = \lambda^2 + \sum_{I \backslash I_M} f_I^2 \tag{21}$$

Since the system is positive definite, we know that even for more than two motions equation (20) is the only possible solution of the system (19). For up to four motions, the motion parameters can be obtained from the mixed-motion parameters analytically as described above since the increased number of motions will just increase the order of the polynomial. For more than four motions, the roots of the polynomial $Q(z)$ can be found by numerical methods.

4. SEPARATION OF MOTION LAYERS

In Section 3 we have shown how to estimate the motion fields in case of transparent motions. We will now show how to use them to separate the spectra of the overlaid images.

Firstly, we discretize time to $t_k = k\Delta t$, $k = 0, \ldots$ and transform the discretized motion model in (9), i.e.,

$$f_{t_k}(\mathbf{x}) = f(\mathbf{x}, t_k) = g_1(\mathbf{x} - t_k\mathbf{v}_1) + g_2(\mathbf{x} - t_k\mathbf{v}_2) + \cdots + g_N(\mathbf{x} - t_k\mathbf{v}_N) \tag{22}$$

to the frequency domain by use of the Fourier transform. Such, we obtain the following system of equations

$$F_{t_k}(\omega) = \phi_1^k G_1(\omega) + \phi_2^k G_2(\omega) + \cdots + \phi_N^k G_N(\omega), \qquad k = 0, \ldots, \tag{23}$$

where $\phi_n = e^{j\omega\cdot\mathbf{v}_n\Delta t}$, $n = 1, \ldots, N$, $\omega = (\omega_x, \omega_y)$ is the frequency variable and uppercase letters are the Fourier transform of the respective lower case letters, e.g., F_{t_k} is the Fourier transform of f_{t_k}.

Secondly, we note that the first N equations in the system (23) can be written in matrix form as

$$\begin{pmatrix} F_{t_0} \\ F_{t_1} \\ \vdots \\ F_{t_{N-1}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \cdots & \phi_N \\ \vdots & & & \vdots \\ \phi_1^{N-1} & \phi_2^{N-1} & \cdots & \phi_N^{N-1} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} \tag{24}$$

or in short notation $\mathbf{F}_{t_0} = \mathbf{B}_N\mathbf{G}$, where $\mathbf{F}_{t_0} = (F_{t_0}, \ldots, F_{t_{N-1}})$, $\mathbf{G} = (G_1, \ldots, G_N)$ and $\mathbf{B}_N$ is the above matrix containing the phase terms. $\mathbf{B}_N$ is a Vandermonde matrix that is invertible if its entries $\phi_1, \ldots, \phi_N$ are all distinct. The separation is, therefore, possible by using N successive frames of the image sequence by inversion of $\mathbf{B}_N$, i.e.,

$$\mathbf{G} = \mathbf{B}_N^{-1}\mathbf{F}_{t_0}. \tag{25}$$

Note, however, that the separation is not possible at all frequencies. The problematic frequencies are those where two or more phase values are identical because the rank of the matrix B_N is then reduced. This is an important observation because it defines the support where multiple phases can occur by the following equation:

$$\phi_m = \phi_n \iff e^{j(\mathbf{v}_m - \mathbf{v}_n)\cdot\omega\Delta t} = 1 \iff (\mathbf{v}_m - \mathbf{v}_n)\cdot\omega = 2k\pi, \qquad k = 0, \ldots \tag{26}$$

On the above defined lines, the Fourier transforms of at least two transparent layers cannot be separated. If the difference vectors $\mathbf{v}_m - \mathbf{v}_n$, $m, n = 1, \ldots, N$ are not aligned, we have a number of $\binom{N}{2}$ lines with inseparable frequency components going through the origin of frequency space. A possible solution would be to interpolate the values on these lines from the neighboring frequency values of the separated layers.

5. PHASE-BASED MODEL FOR MULTIPLE TRANSPARENT MOTIONS

Frequency-domain based approaches to transparent motions are based on the observation that motion induces a phase shift. This relationship has been used for the estimation of only one motion by Jepson and Fleet.[7] For transparent motions, this translates to Equation (23) that we repeat here as an aid:

$$F_{t_k}(\omega) = \phi_1^k G_1(\omega) + \phi_2^k G_2(\omega) + \cdots + \phi_N^k G_N(\omega), \qquad k = 0, \ldots \tag{27}$$

Equation (27) has been analytically solved by Vernon[3] to obtain the phase components and to separate the layers for the simplest case of only two motions. In the following we will solve (27) for the general case of N motions. Note that we already have the solution for the Fourier transform of the layers in terms of the phase shifts, see Equation (25). To obtain the phase shifts, we first simplify notation by setting $\Phi_k = (\phi_1^k, \ldots, \phi_N^k)$ and $\mathbf{G} = (G_1, \ldots, G_N)$. We then obtain the following expressions for the above system :

$$F_{t_k} = \Phi_k \cdot \mathbf{G}, \qquad k = 0, \ldots \tag{28}$$

Our goal now is to obtain the phase-components vector $\Phi_1 = (\phi_1, \ldots, \phi_N)$ by cancellation of the unknown Fourier-transforms vector $\mathbf{G}$ of the image layers in the system above. Given $N+1$ images of the sequence, we define the polynomial

$$p(z) = (z - \phi_1) \cdots (z - \phi_N) = z^N + a_1 z^{N-1} + \cdots + a_N \tag{29}$$

with unknown coefficients $a_1, \ldots, a_N$. Now the phase terms $\phi_1, \ldots, \phi_N$ are the roots of $p(z)$, i.e., $p(\phi_n) = 0$, for $n = 1, \ldots, N$ and we observe that

$$\begin{aligned}\Phi_{m+N} + a_1\Phi_{m+N-1} + \cdots + a_N\Phi_m &= (\ldots, \phi_n^{m+N} + a_1\phi_n^{m+N-1} + \cdots + a_N\phi_n^m, \ldots) = \\ &(\ldots, \phi_n^m(\phi_n^N + a_1\phi_n^{N-1} + \cdots + a_N), \ldots) = (\phi_1^m p(\phi_1), \ldots, \phi_n^m p(\phi_n), \ldots, \phi_N^m p(\phi_N)) = \mathbf{0}.\end{aligned} \tag{30}$$

Therefore by inserting (30) in (28) we obtain

$$F_{t_{m+N}} + a_1 F_{t_{m+N-1}} + \cdots + a_N F_{t_m} = (\Phi_{m+N} + a_1\Phi_{m+N-1} + \cdots + a_N\Phi_m) \cdot \mathbf{G} = \mathbf{0} \cdot \mathbf{G} = \mathbf{0} \tag{31}$$

and

$$F_{t_{m+N}} = -a_N F_{t_m} - \cdots - a_1 F_{t_{m+N-1}} \qquad m = 0, \ldots \tag{32}$$

We can use the first N of the above equations to solve for the unknown vector $\mathbf{y} = (a_1, \ldots, a_N)$ of coefficients of $p(z)$ by solving the linear system

$$\begin{pmatrix} F_{t_N} \\ F_{t_{N+1}} \\ \vdots \\ F_{t_{2N-1}} \end{pmatrix} = -\begin{pmatrix} F_{t_0} & F_{t_1} & \cdots & F_{t_{N-1}} \\ F_{t_1} & F_{t_2} & \cdots & F_{t_N} \\ \vdots & \vdots & & \vdots \\ F_{t_{N-1}} & F_{t_N} & \cdots & F_{t_{2N-2}} \end{pmatrix} \begin{pmatrix} a_N \\ a_{N-1} \\ \vdots \\ a_1 \end{pmatrix}. \tag{33}$$

With $\mathbf{F}_N$ for the left hand side vector and $\mathbf{A}_N$ for the matrix in the above system, we obtain the short notation $\mathbf{F}_N = -\mathbf{A}_N\, \mathbf{y}$.

After solving the above system for $a_1, \ldots, a_N$, we obtain the unknown phase changes $\phi_1, \ldots, \phi_N$ as the roots of $p(z)$.

5.1. The case of a singular matrix $\mathbf{A}_N$

The Equations (28) for $\Phi_1 = (\phi_1, \ldots, \phi_N)$ can be solved as long as the matrix $\mathbf{A}_N$ in Equation (33) is not singular. However, as we will show, it is still possible to estimate the phase terms even with the singular matrix. To understand why, we will discuss all the cases in which $\mathbf{A}_N$ is singular. First note that the matrix $\mathbf{A}_N$ nicely factors as

$$\mathbf{A}_N = \begin{pmatrix} F_{t_0} & F_{t_1} & \cdots & F_{t_{N-1}} \\ F_{t_1} & F_{t_2} & \cdots & F_{t_N} \\ \vdots & \vdots & & \vdots \\ F_{t_{N-1}} & F_{t_N} & \cdots & F_{t_{2N-2}} \end{pmatrix} = \mathbf{B}_N \begin{pmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & G_N \end{pmatrix} \mathbf{B}_N^T \tag{34}$$

and since $\mathbf{B}_N$ is a Vandermonde matrix with entries $\phi_1, \ldots, \phi_N$, its determinant is

$$\det \mathbf{A}_N = G_1 \cdots G_N \prod_{m>n} (\phi_m - \phi_n)^2. \tag{35}$$

It follows that there are only two non-exclusive situations where the matrix $\mathbf{A}_N$ can become singular:

1. The Fourier transform of at least one layer vanishes at the frequency ω for which $G_m(\omega) = 0$;
2. Phase shifts are equal, that is, $\phi_m = \phi_n$ for at least one pair m, n.

If two phase shifts are equal. e.g. $\phi_N = \phi_{N-1}$, the number of terms in the sum of Equation (27) reduces to

$$F_{t_k} = \phi_1^k G_1 + \phi_2^k G_2 + \cdots + \phi_{N-1}^k (G_{N-1} + G_N) = \phi_1^k G_1 + \phi_2^k G_2 + \cdots + \phi_{N-1}^k \tilde{G}_{N-1}, \quad k = 0, \ldots, \tag{36}$$

where $\tilde{G}_{N-1} = G_{N-1} + G_N$. Therefore we can suppose without loss of generality that

$$F_{t_k} = \phi_1^k \tilde{G}_1 + \phi_2^k \tilde{G}_2 + \cdots + \phi_R^k \tilde{G}_R, \quad k = 0, \ldots \tag{37}$$

with the above phase shifts $\phi_1, \ldots, \phi_R$ being all distinct and $\tilde{G}_1, \ldots, \tilde{G}_R$ being all different from zero. Note that a phase term could disappear from the equation due to the fact that the coefficients can sum to zero. Since the left hand side of Equation (37) does not change due to factorization and reordering of the terms, we can rewrite Equation (34) as

$$\mathbf{A}_N = \begin{pmatrix} F_{t_0} & F_{t_1} & \cdots & F_{t_{N-1}} \\ F_{t_1} & F_{t_2} & \cdots & F_{t_N} \\ \vdots & \vdots & & \vdots \\ F_{t_{N-1}} & F_{t_N} & \cdots & F_{t_{2N-2}} \end{pmatrix} = \mathbf{B}_N \begin{pmatrix} \tilde{G}_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \tilde{G}_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \tilde{G}_R & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \mathbf{B}_N^T. \tag{38}$$

Therefore $\mathrm{rank}(\mathbf{A}_N) = \mathrm{rank}(\mathbf{A}_R) = R$ and we can compute the phases shifts $\phi_1, \ldots, \phi_R$ by resolving the system

$$\mathbf{F}_R = -\mathbf{A}_R \, \mathbf{y}. \tag{39}$$

5.1.1. Examples for only two motions

We now illustrate the above results for the special case of only two overlaid motions.

1. Rank($\mathbf{A}_2$) = 2: the estimation of the phase ϕ_1, ϕ_2 is possible. First we solve for a_1, a_2 the system $\mathbf{F}_{t_2} = -\mathbf{A}_2\ \mathbf{y}$, that is
$$\begin{pmatrix} F_{t_2} \\ F_{t_3} \end{pmatrix} = -\begin{pmatrix} F_{t_0} & F_{t_1} \\ F_{t_1} & F_{t_2} \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \tag{40}$$
and obtain
$$\begin{aligned} a_1 &= \frac{F_{t_1}F_{t_2} - F_{t_0}F_{t_3}}{F_{t_0}F_{t_2} - F_{t_1}^2} \\ a_2 &= \frac{F_{t_1}^2 - F_{t_0}F_{t_2}}{F_{t_0}F_{t_2} - F_{t_1}^2} \end{aligned} \tag{41}$$
The phases ϕ_1, ϕ_2 are then computed as the roots of
$$p(z) = z^2 + a_1 z + a_2 \tag{42}$$
2. Rank($\mathbf{A}_2$) = 1: the possible cases are $G_1 = 0,\ G_2 \neq 0;\ G_1 \neq 0,\ G_2 = 0$ or $\phi_1 = \phi_2,\ G_1 + G_2 \neq 0$ and we can compute the double phase or one of the two distinct phases from
$$F_{t_1} = F_{t_0}\phi. \tag{43}$$
3. Rank($\mathbf{A}_2$) = 0: in this case $G_1 = G_2 = 0$ or $\phi_1 = \phi_2,\ G_1 + G_2 = 0$ and all equations in (27) degenerate to
$$F_{t_k} = 0, \quad k = 0, \dots \tag{44}$$

6. RESULTS

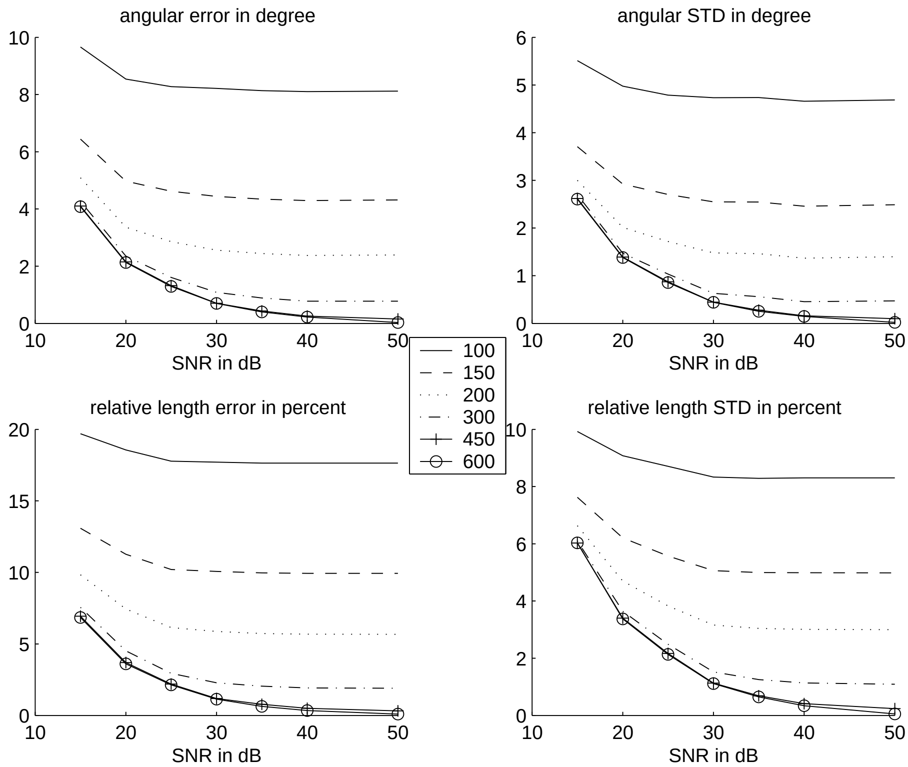
6.1. Derivatives and filters

We computed the derivatives in the spatial domain. The first-order derivatives are Gaussian derivative filter with a sigma of one and size of 7 pixels. The filters have been applied twice to obtain the second-order derivatives.

6.2. Motion estimation

We tested the regularized motion estimation algorithm on real and synthetic images for the case of two motion.

In Figure 1 we analyze robustness against noise. We simulated natural images by generating $1/f$ spectral noise sequences to which we added Gaussian distributed noise at different signal-to-noise ratios (SNR). The errors of the estimated motion vectors are shown as a function of the SNR and the number of iterations. In the first row of Figure 1 we show the absolute mean angular errors and the angular standard derivation (STD) of the estimated motion vectors. The second row displays the errors in relative magnitude and the corresponding STD. Note that we obtain very good result with a SNR of up to 20dB. In all examples the regularization parameter λ had a value of one.



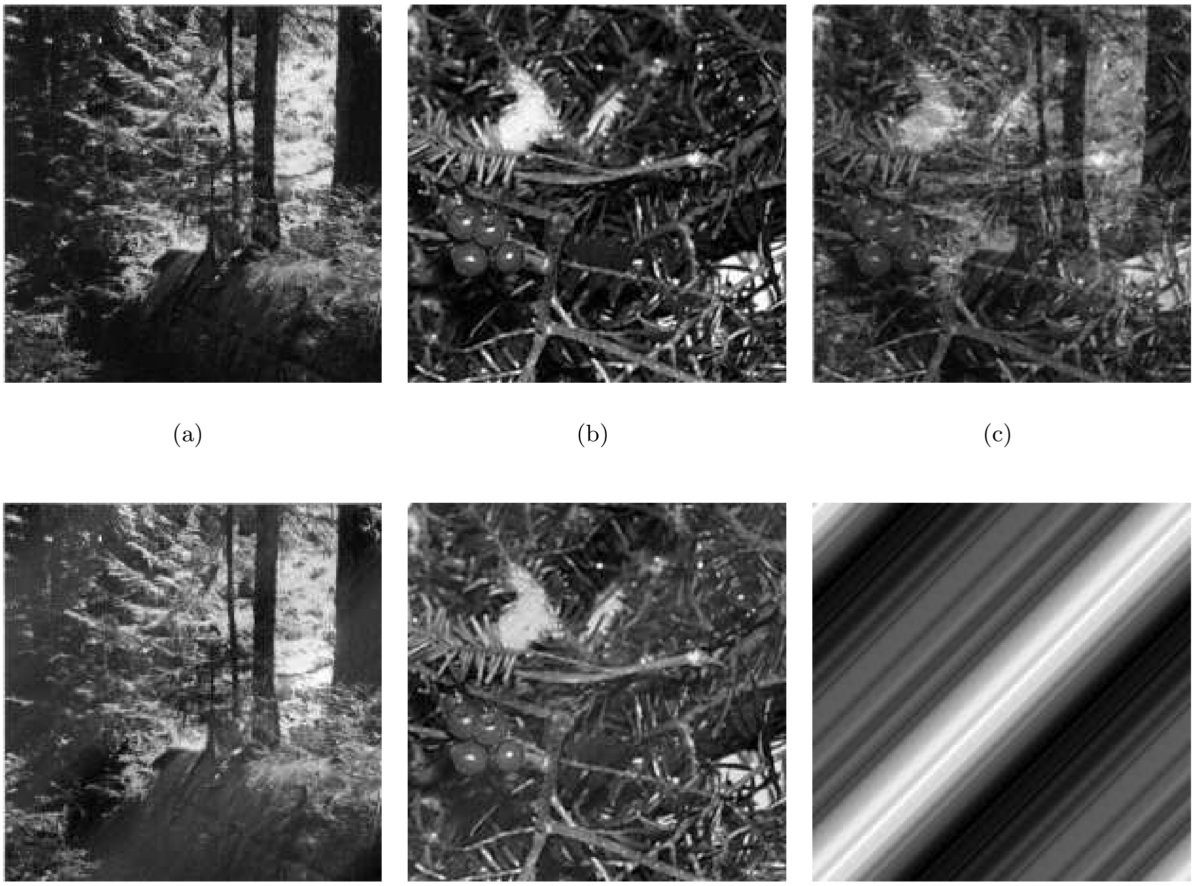
(a)

Figure 1. Performance as a function of noise level and the number of iterations. For details see text.

6.3. Separation of layers

Figure 2 shows an example were the layer separation works very well. In (a) and (b) the original layers are shown, which are then overlaid in (c). In the image sequence the layers were moving with velocities (1,0) and (0,1), respectively. In (d) we show the separation result for the layer (a) and in (e) the result for layer (b). Note that both layers are well detected and that the interpolation errors are not visible in the reconstructed images. However, the errors become visible when subtracting the original in (a) from the separated layer in (d) as shown in (f).

In Figure 3, however, the interpolation errors are evident. In (a) we show the first image of the sequence that is a clown image moving with velocity (1,0) and a cameraman image moving with velocity (0,1). Note that both separated images (c) and (d) are overlaid with an oriented pattern. After subtracting the original image (b) from the separation layer (e) we obtained the difference image (f) that shows the oriented error patterns. Obviously, the visibility of the errors is due to missing texture. However, the errors could be reduces with a better interpolation method. We have computed the missing frequency values as the average of the neighboring values. In Figure 4 we show the results of motion estimation with a realistic sequence and demonstrate the performance of our layer-separation algorithm for reflection removal. In (a) we show one frame of the original Mona Lisa sequence. We used the estimated motion vectors for nulling one of the two motions out of the sequence by computing $\alpha(\mathbf{u})f$ and $\alpha(\mathbf{v})f$. The results are shown in (c) and (d) respectively. In (b) we show the Mona Lisa layer only as the result of the separation algorithm. Note that the reflections are removed.



(a) (b) (c)

(d) (e) (f)

Figure 2. Separation of layers for high-textured images. For details see text.

7. DISCUSSION

We have presented a novel method for the estimation of multiple transparent motions that is based on an iterative solution of a linear system of equations. The system is obtained by introducing a regularization term for the mixed-motion parameters. The motion-vector components are then obtained from the mixed-motion parameters by solving for the roots of a complex polynomial. Alternative regularization procedures could be used, since we have succeeded to linearize the problem of determining multiple overlaid motions. Due to the linearization, it is possible to incorporate regularization and deal with more that two motions. We have presented good results on synthetic and real images. However, the way we have computed the partial derivatives still needs to be optimized and filters other than derivatives could be used, as outlined in [5], to increase robustness. Possible extensions to cope with occluded motions have been proposed in [8]. In addition to an regularized estimation of multiple motions we have generalized a method based on the phase in the Fourier transform. We have used this method to separate the motion layers and have shown some inherent limits of the method. Further research will deal with alternative methods for the separation of the motion layers that avoid these problems.

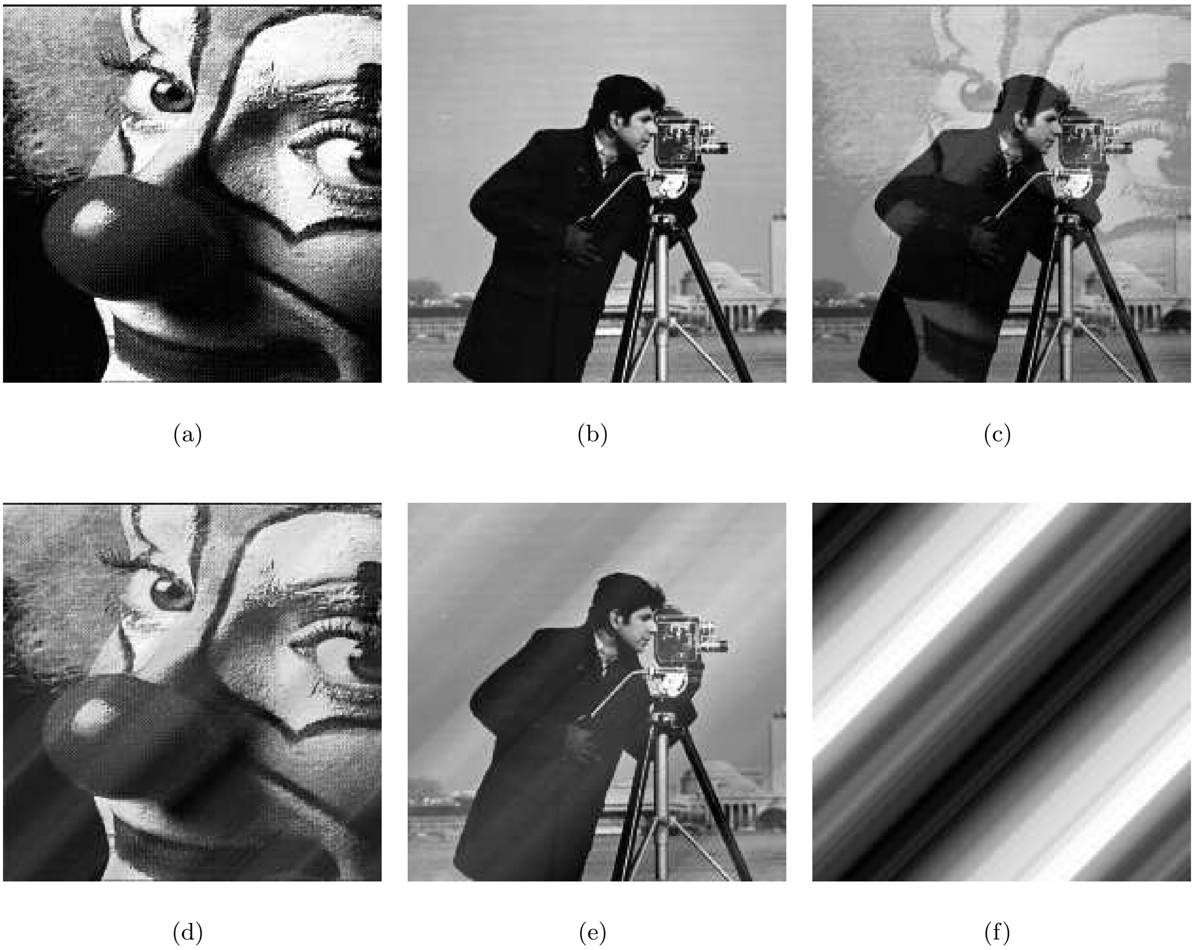


Figure 3. Separation of layers for low-textured images. For details see text.

ACKNOWLEDGMENTS

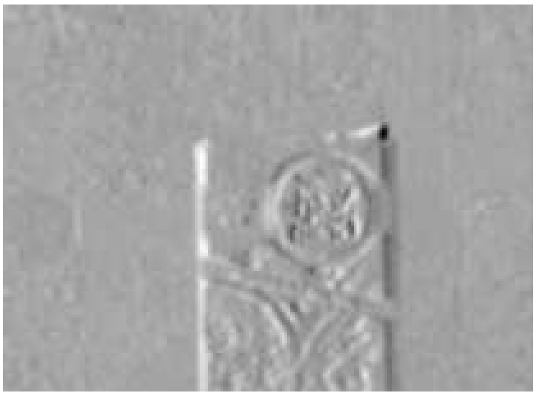
Work is supported by the *Deutsche Forschungsgemeinschaft* under Ba 1176/7-1.

REFERENCES

1. M. J. Black and P. Anandan, "The robust estimation of multiple motions: parametric and piecewise-smooth flow fields," *Computer Vision and Image Understanding* **63**, pp. 75–104, Jan. 1996.
2. M. Shizawa and K. Mase, "Simultaneous multiple optical flow estimation," in *IEEE Conf. on Computer Vision and Pattern Recognition*, **I**, pp. 274–8, IEEE Computer Soc., (Atlantic City, NJ), June 1990.
3. D. Vernon, "Decoupling Fourier components of dynamic image sequences: a theory of signal separation, image segmentation and optical flow estimation," in *Computer Vision - ECCV'98: 5th European Conf. on Computer Vision*, H. Burkhardt and B. Neumann, eds., *LNCS* **1407/II**, pp. 68–85, Springer Verlag, Jan. 1998.
4. J. Y. A. Wang and E. H. Adelson, "Representing moving images with layers," *IEEE Transactions on Image Processing* **3**(5), pp. 625–38, 1994.

(a)

(b)



(c)

(d)

Figure 4. Separation of reflectance components. For details see text.

5. C. Mota, I. Stuke, and E. Barth, “Analytic solutions for multiple motions,” in *Proc. IEEE Int. Conf. on Image Processing*, **II**, pp. 917–20, IEEE Signal Processing Soc., (Thessaloniki, Greece), Oct. 7–10, 2001.
6. B. Horn and B. Schunck, “Determining optical flow,” *Artificial Intelligence* **17**, pp. 185–203, Aug. 1981.
7. D. J. Fleet and A. D. Jepson, “Computation of component image velocity from local phase information,” *International Journal of Computer Vision* **5**(1), pp. 77–104, 1990.
8. E. Barth, I. Stuke, and C. Mota, “Analysis of motion and curvature in image sequences,” in *Proc. 5th Southwest Symp. on Image Analysis and Interpretation*, pp. 206–10, IEEE Computer Soc., (Santa Fe, NM), Apr. 7–9, 2002.