[5] P. Moulin, "A new look at signal-adapted QMF bank design," in *Proc. ICASSP*, May 1995.
[6] S. Ohno and H. Sakai, "Optimization of filter banks using cyclostationary spectral analysis," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, Detroit, MI, May 1995, pp. 1292–1295.
[7] M. J. T. Smith and T. P. Barnwell, III, "A unifying framework for analysis/synthesis systems based on maximally decimated filter banks," in *Proc. IEEE Int. Conf. Acoust. Speech, Signal Process.*, Tampa, FL, Mar. 1985, pp. 521–524.
[8] D. Taubman and A. Zakhor, "A multi-start algorithm for signal-adaptive subband systems," in *Proc. ICASSP*, 1992, pp. 213–216.
[9] A. Tirakis, A. Delopoulos, and S. Kollias, "2-D filter bank design for optimal reconstruction using limited subband information," *IEEE Trans. Image Processing*, vol. 4, pp. 1160–1165, Aug. 1995.
[10] M. Tsatsatsanis and G. Giannakis, "Principal component filter banks for optimal multiresolution analysis," *IEEE Trans. Signal Processing*, vol. 43, pp. 1766–1777, Aug. 1995.
[11] M. Unser, "On the optimality of ideal filters for pyramid and wavelet signal approximation," *IEEE Trans. Signal Processing*, vol. 41, pp. 3591–3596, Dec. 1993.
[12] B. Usevitch and M. Orchard, "Smooth wavelets, transform coding, and Markov-1 processes," in *Proc. ISCAS*, 1993, pp. 527–530.
[13] P. P. Vaidyanathan, "Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 476–492, Apr. 1987.
[14] ——, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[15] ——, "Theory of optimal orthonormal filter banks," in *Proc. ICASSP*, 1996.
[16] L. Vandendorpe, "CQF filter banks matched to signal statistics," *Signal Process.*, vol. 29, pp. 237–249, 1992.
[17] M. Vetterli, "Splitting a signal into subsampled channels allowing perfect reconstruction," in *Proc. IASTED Conf. Appl. Signal Process. Digital Filtering*, Paris, France, June 1985.
[18] E. Viscito and J. Allebach, "The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices," *IEEE Trans. Circuits Syst.*, vol. 38, Jan. 1991.
[19] B. Xuan and R. Bamberger, "Multi-dimensional, paraunitary principal component filter banks," in *Proc. IEEE ICASSP*, Detroit, MI, May 1995, pp. 1488–1491.
[20] B. Xuan, "Multi-dimensional, paraunitary principal component filter banks," Ph.D. dissertation, Washington State Univ., Pullman, WA, Aug. 1996.

Subspace Approach for the Design of Cosine-Modulated Filter Banks with Linear-Phase Prototype Filter

Alfred Mertins

***Abstract*— In this correspondence, a method for designing perfect reconstruction (PR) prototypes for paraunitary cosine-modulated filter banks is presented. The design procedure is based on a subspace approach that allows linear combinations of even-length linear-phase PR prototype filters in such a way that the resulting filter is also a linear-phase PR prototype. Within a given subspace, the weights of the optimal linear combination can easily be computed via an eigenanalysis. The filter design is carried out iteratively while the PR property is guaranteed throughout the design process. No nonlinear optimization routine is needed. As a special case, the proposed approach allows the design of discrete-coefficient prototypes, which are of great interest for efficient hardware implementations.**

***Index Terms*— Cosine modulation, discrete coefficients, filter banks, integer coefficients.**

I. Introduction

Cosine-modulated filter banks are very popular in signal processing because of their efficiency [1]–[9]. In this class of filter banks, all analysis and synthesis filters are modulated versions of a single prototype. The implementation of the complete filter bank depicted in Fig. 1 only requires the implementation of polyphase components of the prototype and of the modulation, which itself can be efficiently realized via FFT's. Cosine-modulated filter banks can be designed as pseudo QMF banks [2], paraunitary filter banks [3]–[6], and as biorthogonal filter banks allowing a low reconstruction delay [7]–[10]. In this correspondence, only linear-phase prototypes are considered.

The quality of the filter bank for a given application mainly depends on the properties of the prototype. For the design of the prototype, which will be denoted as $P(z)$, we can follow various strategies. A method that structurally guarantees the perfect reconstruction (PR) property of the filter bank is the use of lattice factorizations [11]. For this method, a good starting point is required because we have to optimize angles in a cascade of lattices, and the relations between the angles and the impulse response are highly nonlinear. A second method that is typically less sensitive to the starting point is the quadratic-constraint algorithm [12]. This method does not inherently guarantee PR, but the PR requirements can be satisfied with arbitrary accuracy. A simple but efficient iterative design method for practically useful near PR prototypes was presented in [13]. This method is based on the older pseudo-QMF ideas [2] rather than on the PR constraints [3]–[5] so that on principle, only near PR prototypes can be designed. A further method for designing near PR prototypes was proposed in [14].

In this correspondence, a new design method is presented that, like the lattice factorization, guarantees the PR property. The optimization is performed iteratively by optimizing linear combinations of impulse responses within suitable linear subspaces. Throughout the

Manuscript received July 8, 1997; revised March 19, 1998. The associate editor coordinating the review of this paper and approving it for publication was Dr. Xiang-Gen Xia.

The author is with the Department of Electrical and Electronic Engineering, The University of Western Australia, Nedlands, Australia (e-mail: mertins@ee.uwa.edu.au).

Publisher Item Identifier S 1053-587X(98)07081-0.

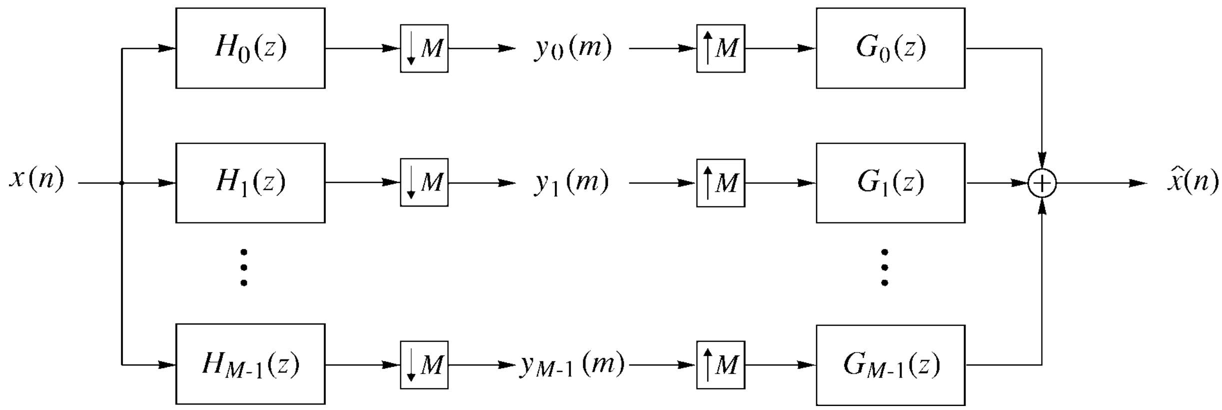

Fig. 1. Critically subsampled M-channel filter bank.

filter design process, no nonlinear optimization routine is required. However, nonlinear optimization may be used in order to achieve further improvements.

Filters with integer-valued coefficients are quite desirable because they allow the efficient implementation of filter banks. The simplest way to design such an integer prototype is to quantize the coefficients of a given prototype. Clearly, when doing this, the PR property gets lost, and we will need relatively many bits in order to achieve at least an almost PR. An alternative is to realize the filter bank in a lattice structure, which allows us to achieve PR of integer input signals within a discrete implementation [15]. In this correspondence, a different method is proposed that keeps the PR property throughout the design process while dealing only with integers.

II. Paraunitary Cosine-Modulated Filter Banks

In this correspondence, we regard the case where the number of channels M and the filter length L are even. Furthermore, critical subsampling and a real-valued lowpass prototype $p(n)$ are considered.

If $p(n)$ satisfies the symmetry condition $p(n) = p(L-n-1)$, $n = 0, 1, \cdots, L/2-1$ and if its polyphase components $P_k(z)$ satisfy the condition

$$\tilde{P}_k(z)\,P_k(z) + \tilde{P}_{M+k}(z)\,P_{M+k}(z) = 1$$
$$k = 0, 1, \cdots, \frac{M}{2} - 1 \tag{1}$$

then the filter bank is paraunitary, that is, the filter bank has the PR property, and moreover, it provides a unitary transform [3]–[5], [11]. Herein, the polyphase components $P_k(z)$ are defined as $P_k(z) = \sum_{n=0}^{m-1} z^{-n} p_k(n)$, where $p_k(n) = p(2nM + k)$, $k = 0, 1, \cdots, 2M - 1$. Terms with a tilde accent in (1) denote $\tilde{P}_k(z) = P_k(z^{-1})$.

When a prototype has a high stopband attenuation, it will give good performance in a wide range of applications. Therefore, the classical way to measure the quality of a prototype filter is to measure the stopband attenuation or stopband energy. Here, we follow this classical idea and formulate the optimization problem using the Rayleigh quotient

$$C(\boldsymbol{p}) = \frac{\boldsymbol{p}^T \boldsymbol{V}_s \boldsymbol{p}}{\boldsymbol{p}^T \boldsymbol{p}} \overset{!}{=} \min. \tag{2}$$

The vector $\boldsymbol{p}$ contains the unknown filter coefficients $\boldsymbol{p} = [p(0), p(1), \cdots, p(L-1)]^T$, and $\boldsymbol{V}_s$ is a weighting matrix defined by

$$\boldsymbol{p}^T \boldsymbol{V}_s \boldsymbol{p} = \int_{\text{stopband}} G_s(\omega) |P(\omega)|^2 \, d\omega \tag{3}$$

where $G_s(\omega)$ is a non-negative weighting function, and $P(\omega)$ is the Fourier transform of $p(n)$. For $G_s(\omega) = 1$ within the stopband, the criterion (2) states that we are seeking filters with minimum stopband energy under the restriction that the energy of the prototype is fixed. Unfortunately, the optimal solution to (2) will not satisfy condition (1); therefore, (1) must be included in the optimization process.

III. Subspace Approach

Let us consider the optimality criterion (2), and let us assume that we have a set of basis vectors for the design of our optimal prototype $p(n)$, that is, let us assume that we can write $\boldsymbol{p}$ in the following form, where the matrix $\boldsymbol{F}$ contains the basis, and α contains the coefficients to be optimized: $\boldsymbol{p} = \boldsymbol{F}\boldsymbol{\alpha}$. If all linear combinations of the columns of matrix $\boldsymbol{F}$ lead to a PR prototype $\boldsymbol{p}$, we can formulate the optimization problem as

$$C(\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}^T \boldsymbol{U}_s \boldsymbol{\alpha}}{\boldsymbol{\alpha}^T \boldsymbol{U}_p \boldsymbol{\alpha}} \overset{!}{=} \min, \quad \text{where} \quad \boldsymbol{U}_s = \boldsymbol{F}^T \boldsymbol{V}_s \boldsymbol{F}$$
$$\boldsymbol{U}_p = \boldsymbol{F}^T \boldsymbol{F} \tag{4}$$

and we can solve (4) for the optimal $\boldsymbol{\alpha}$ in an unrestricted way. Thus, the solution is given by the eigenvector α corresponding to the minimum eigenvalue λ of the generalized eigenvalue problem

$$\boldsymbol{U}_s \boldsymbol{\alpha} = \lambda \boldsymbol{U}_p \boldsymbol{\alpha}. \tag{5}$$

Let us now focus on the linear combination of two prototype filters $A(z)$ and $B(z)$

$$P(z) = \alpha_1\, A(z) + \alpha_2\, B(z). \tag{6}$$

Here, we have

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} a(0) & a(1) & \cdots & a(L-1) \\ b(0) & b(1) & \cdots & b(L-1) \end{bmatrix}^T \tag{7}$$

where $a(n)$ and $b(n)$ are the impulse responses of the systems $A(z)$ and $B(z)$, respectively. In order to allow the subspace approach, let us require that for all $\alpha_1, \alpha_2 \in \mathbb{R}$ (except for $\alpha_1 = \alpha_2 = 0$), the resulting filter $P(z)$ satisfies the PR condition (1) up to some scaling factor γ, which has to be independent of k. This means

$$\tilde{P}_k(z) P_k(z) + \tilde{P}_{M+k}(z) P_{M+k}(z) = \gamma, \quad \gamma \neq 0$$
$$k = 0, 1, \cdots, \frac{M}{2} - 1 \tag{8}$$

where

$$P_k(z) = \alpha_1\, A_k(z) + \alpha_2\, B_k(z). \tag{9}$$

Since the choices $[\alpha_1, \alpha_2] = [1, 0]$ and $[\alpha_1, \alpha_2] = [0, 1]$ are valid, the filters $A(z)$ and $B(z)$ must satisfy the PR condition (1) at least up to a scaling factor in order to allow $P(z)$ to fulfill (8). Scaling of $A(z)$ and $B(z)$ does not change the subspace spanned by these filters; therefore, we may require that the PR conditions

$$\tilde{A}_k(z)A_k(z) + \tilde{A}_{M+k}(z)A_{M+k}(z) = 1 \tag{10}$$

$$\tilde{B}_k(z)B_k(z) + \tilde{B}_{M+k}(z)B_{M+k}(z) = 1 \tag{11}$$

are satisfied for $k = 0, 1, \cdots, M/2 - 1$. Given the filters $A(z)$ and $B(z)$, we can look for the best linear combination of these filters in the sense of (4). As we will see below, we cannot find the global optimum this way, but we can use this procedure iteratively. We will return to this point in Section IV.

Construction of $B_k(z)$: Let $A_k(z)$ be the polyphase components of a given prototype that satisfies (10). In order to construct filters $B_k(z)$ in such a way that (8) and (11) are satisfied, we combine (8) and (9). This leads to

$$\begin{aligned}&\tilde{A}_k(z)\,B_k(z) + A_k(z)\tilde{B}_k(z) + \tilde{A}_{M+k}(z)\,B_{M+k}(z)\\ &\quad + A_{M+k}(z)\,\tilde{B}_{M+k}(z) = c = \text{const.}\end{aligned} \tag{12}$$

for $k = 0, 1, \cdots, M/2 - 1$, where $c = (\gamma - \alpha_1^2 - \alpha_2^2)/(\alpha_1\alpha_2)$. Given $A_k(z)$ and $A_{M+k}(z)$, (12) is nothing but an underdetermined linear set of equations for $B_k(z)$ and $B_{M+k}(z)$. This means that we can choose any solution to (12) for $c \neq 0$ and add any further solution from the nullspace ($c = 0$). Since $B_k(z) = A_k(z)$ is a simple (but valid) solution to (12), it becomes clear that we should look for solutions in the nullspace only

$$\begin{aligned}&\tilde{A}_k(z)\,B_k(z) + A_k(z)\,\tilde{B}_k(z) + \tilde{A}_{M+k}(z)\,B_{M+k}(z)\\ &+ A_{M+k}(z)\,\tilde{B}_{M+k}(z) = 0, \quad k = 0, 1, \cdots, \frac{M}{2} - 1.\end{aligned} \tag{13}$$

Equation (13) can be interpreted as a set of orthogonality relations. Note that (13) also implies orthogonality of the complete impulse responses $a(n)$ and $b(n)$ in the classical sense

$$\sum_{n=0}^{L-1} a(n)b(n) = 0. \tag{14}$$

Filters $B_k(z)$ satisfying (13) are

$$\begin{aligned}B_k(z) &= C_k(z)\,A_{M+k}(z)\\ B_{M+k}(z) &= -\,\tilde{C}_k(z)\,A_k(z)\end{aligned} \tag{15}$$

and

$$\begin{aligned}B_k(z) &= D_k(z)\,\tilde{A}_{M+k}(z)\\ B_{M+k}(z) &= -\,D_k(z)\,\tilde{A}_k(z).\end{aligned} \tag{16}$$

Among these solutions, we are interested in finding the ones that also satisfy (11) and lead to linear-phase filters $B(z)$. These requirements restrict the systems $C_k(z)$ and $D_k(z)$ to be simple delays. Thus, we have solutions of the form

$$\begin{aligned}&B_k(z) = \pm\, z^{-\ell_k} A_{M+k}(z), \quad B_{M+k}(z) = \mp z^{\ell_k} A_k(z)\\ &\qquad k = 0, 1, \cdots, \frac{M}{2} - 1\end{aligned} \tag{17}$$

and

$$\begin{aligned}B_k(z) &= \pm\, z^{-\ell_k} z^{-(m-1)}\,\tilde{A}_{M+k}(z)\\ B_{M+k}(z) &= \mp\, z^{-\ell_k} z^{-(m-1)}\,\tilde{A}_k(z)\\ &\qquad k = 0, 1, \cdots, \frac{M}{2} - 1.\end{aligned} \tag{18}$$

By using the linear-phase property of the prototypes, the remaining polyphase filters are derived as $B_k(z) = z^{-(m-1)}\tilde{B}_{2M-1-k}(z)$.

Dimensionality of the Subspace: We will now show that the subspace approach only allows the construction of 2-D subspaces with the property that any linear combination of the elements of the subspace yields a PR prototype. For this, let us evaluate (12) for two filters $U(z)$ and $V(z)$ [instead of $A(z)$ and $B(z)$], where $U(z)$ and $V(z)$ are taken from (17) and (18). We have two cases:

- $U_k(z)$ and $V_k(z)$ are taken either from (17) or from (18). Then, the expression (12) becomes $\pm(z^{\ell_{k_u} - \ell_{k_v}} + z^{\ell_{k_v} - \ell_{k_u}})$, where ℓ_{k_u} and ℓ_{k_v} are the delays used in the construction of $U_k(z)$ and $V_k(z)$. Thus, (12) can only be satisfied for $U(z) = \pm V(z)$.
- $U_k(z)$ is taken from (17), and $V_k(z)$ is taken from (18) (or vice versa). Evaluating the left-hand side of (12) then yields

$$\begin{aligned}&z^{\ell_{k_u} - \ell_{k_v} - m + 1}\tilde{A}^2_{M+k}(z) + z^{\ell_{k_v} - \ell_{k_u} + m - 1}A^2_{M+k}(z)\\ &\quad + z^{-\ell_{k_u} - \ell_{k_v} - m + 1}\tilde{A}^2_k(z) + z^{\ell_{k_u} + \ell_{k_v} + m - 1}A^2_k(z).\end{aligned}$$

For a nontrivial FIR filter $A(z)$, this expression cannot become a constant.

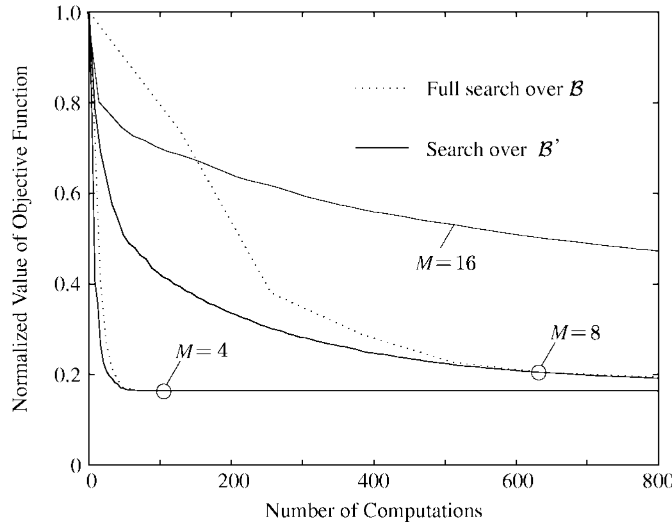

Fig. 2. Convergence properties of a full search over $\mathcal{B}$ and a reduced search over $\mathcal{B}'$. For $M = 16$, the full search over $\mathcal{B}$ is not shown in the plot because one iteration step already requires 32 768 computations. The values of the objective function are normalized on their initial values. As initial filter, the ELT prototype [17] was used.

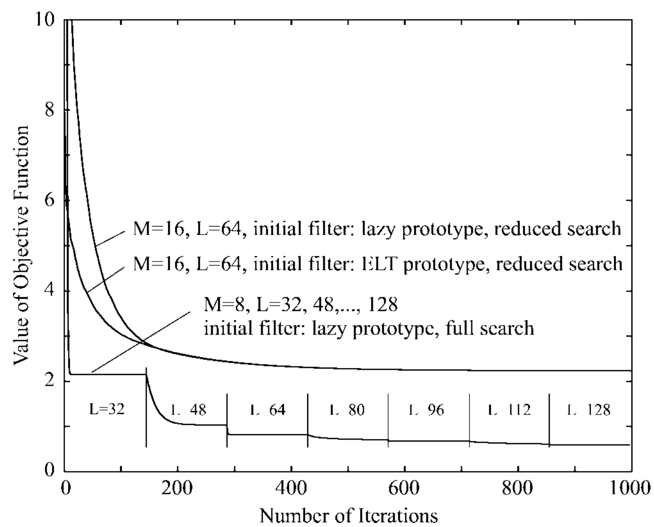

Fig. 3. Convergence properties. For $M = 8$, the filter length has been increased every 142 iterations by 16 taps. The initial values of the objective function are as follows. $M = 16$, lazy prototype: 234.33; $M = 8$, ELT: 7.94; $M = 8$, lazy prototype: 230.7.

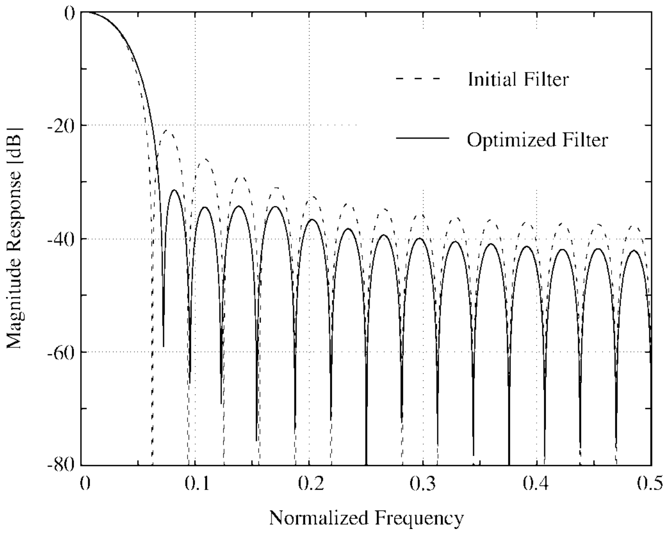

Fig. 4. Magnitude frequency responses of eight-channel filter banks with 32-tap filters.

TABLE I
QUALITY MEASURE $C(\boldsymbol{p})$ FOR THE PROPOSED METHOD AND THE QCLS METHOD FROM [12]. PARAMETERS: $\omega_s = \pi/M; G_s(\omega) = 256/2\pi$

M	4	4	8	8	16	16
L	16	32	32	64	64	128
proposed	2.047	0.850	2.143	0.815	2.233	1.071
QCLS	2.047	0.073	2.145	0.082	2.237	0.109

TABLE II
PERFECT RECONSTRUCTION PROTOTYPES FOR FOUR-BAND FILTER BANKS WITH INTEGER COEFFICIENTS

	$p(n)$					
n	(a)	(b)	(c)	(d)	(e)	(f)
0			-1	-14	-98	-168
1			0	-6	-59	-89
2			0	7	83	118
3			2	33	231	396
4		1	4	56	630	910
5		1	6	96	944	1424
6	1	2	7	112	1328	1888
7	1	2	8	132	1485	2145
$C(\boldsymbol{p})$	221	37.5	7.0	4.2	3.4	2.2

An extension of the subspace approach from two to three dimensions requires the existence of an additional filter that satisfies (12) with respect to both $A(z)$ and $B(z)$. Since such a filter must belong to the class defined by (17) and (18), and since two filters constructed via (17) and (18) cannot satisfy (12), it turns out that such an extension is impossible. The restriction to 2-D spaces also shows that the subspace method cannot be complete. Thus, we cannot find a set of basis vectors where all linear combinations of these vectors yield PR prototypes (up to some scaling factor) and where all PR prototypes satisfying (1) can be written as linear combinations of these vectors.

Properties of the Solutions for $B(z)$: In the construction of our polyphase filters $B_k(z)$, we have the choice to take the solution from (17) or from (18), we can choose ℓ_k, and we can also choose the signs. This means that (17) and (18) define an infinite number of impulse responses $b(n)$. The most important ones are those for $\ell_k = 0$ because then, the filter $B(z)$ has the same support as $A(z)$. When optimizing a prototype and having an initial filter $A(z)$ that already has the final length, $\ell_k = 0$ will be the only choice. A choice $\ell_k \neq 0$ allows us to increase the filter length, which is useful when the initial filter is shorter than the final one. However, since the filter length rapidly

TABLE III
PERFECT RECONSTRUCTION PROTOTYPES FOR EIGHT-BAND FILTER BANKS WITH INTEGER COEFFICIENTS

	$p(n)$					
n	(a)	(b)	(c)	(d)	(e)	(f)
0			-1	-6	-72	-2190
1			-1	-4	-97	-1901
2			0	0	-41	-1681
3			0	-6	-48	-426
4			0	7	56	497
5			0	0	62	2542
6			2	8	194	3802
7			2	17	204	6205
8		1	4	24	390	9678
9		1	4	33	524	13197
10		1	6	41	656	16359
11		1	6	48	774	19398
12	1	2	7	56	903	22631
13	1	2	7	62	992	24738
14	1	2	8	66	1048	26394
15	1	2	8	68	1105	27421
$C(\boldsymbol{p})$	230.7	43.7	12.0	6.7	3.7	2.4

TABLE IV
PERFECT RECONSTRUCTION PROTOTYPES FOR 16-BAND FILTER BANKS WITH INTEGER COEFFICIENTS

	$p(n)$					
n	(a)	(b)	(c)	(d)	(e)	(f)
0			-1	-6	-79	-2806
1			-1	-6	-72	-2508
2			-1	-4	-97	-2910
3			-1	-4	-72	-2180
4			-1	-4	-72	-2428
5			0	0	-41	-1374
6			0	-6	-48	-1480
7			0	0	0	0
8			0	0	0	0
9			0	7	56	1840
10			0	0	62	1943
11			2	8	124	3896
12			2	8	124	4040
13			2	8	194	6305
14			2	17	204	6456
15			2	17	278	8817
16		1	4	24	316	11224
17		1	4	24	390	13585
18		1	4	33	524	15720
19		1	4	33	594	17985
20		1	4	33	594	20031
21		1	6	41	656	21984
22		1	6	48	774	23865
23		1	6	55	830	25705
24	1	2	7	50	855	28190
25	1	2	7	56	903	29670
26	1	2	7	62	992	31088
27	1	2	8	66	1023	32142
28	1	2	8	66	1023	33330
29	1	2	8	66	1048	34060
30	1	2	8	68	1105	34970
31	1	2	8	68	1112	35268
$C(\boldsymbol{p})$	234.3	45.5	12.58	7.0	3.7	2.7

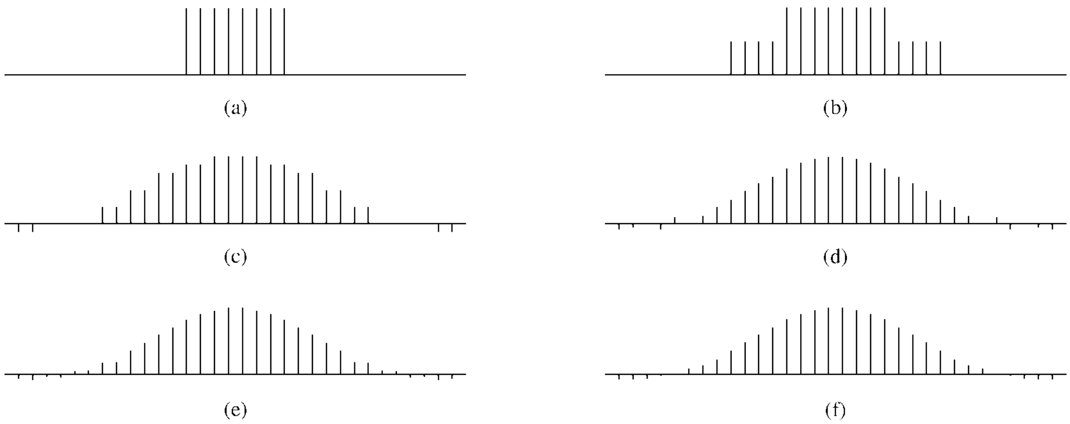

Fig. 5. Impulse responses of the eight-channel prototypes from Table II.

increases when repeatedly using optimization steps with $\ell_k \neq 0$, the optimization will typically involve several steps with $\ell_k = 0$.

For the choice $\ell_k = 0$, the total number of impulse responses $b(n)$ defined by (17) and (18) is 2^M. One half of this set can be generated from the other half by a simple sign change so that we can maximally expect 2^{M-1} filters $b(n)$ with different frequency responses. Such a set of 2^{M-1} filters will be denoted as $\mathcal{B}$. All elements of $\mathcal{B}$ satisfy the PR condition (11), and they are orthogonal to $a(n)$ in the sense of (13). During optimization, we will have to find the filter $b(n)$, which [together with the given $a(n)$] leads to the best filter $P(z)$. In order to reduce the number of filters that have to be tested, it is useful to find a linearly independent subset of $\mathcal{B}$, which is denoted as $\mathcal{B}'$, and to test only the filters in $\mathcal{B}'$. Such a subset $\mathcal{B}'$ is given in (19), shown at the bottom of this page. Each row vector stands for a possible collection $[B_0(z), B_1(z), B_2(z), B_3(z), \cdots]$. It is easily verified that no further linearly independent solution exists for $\ell_k = 0$ so that the total number of linearly independent solutions is M. If different choices are considered for ℓ_k, the number increases accordingly.

IV. Optimization Procedure, Convergence, and Results

In this section, the design of prototypes with infinite-precision coefficients is presented. Modifications that allow the design of discrete-coefficient prototypes will be discussed in Section V.

The filter-design method consists of the following steps:

1) Given a PR prototype $a(n)$, we construct the set of filters $b(n)$ from (17) and (18). Alternatively, we construct a subset that contains linearly independent filters.
2) For all filters designed in Step 1, we solve the optimization problem (4) with $\boldsymbol{F}$ and α according to (7). Then, we select the best candidate. Note that due to (10), (11), and the orthogonality relation (14), $\boldsymbol{U}_p$ becomes the M-fold of the identity matrix. Further note that the eigenvalue problem (5), which gives the solution to (4), only contains matrices of size 2×2 so that simple analytical solutions for the eigenvalues and eigenvectors can be provided.
3) The optimal linear combination of $a(n)$ and the selected $b(n)$ is taken as a new initial solution for Step 1. The process is continued until the incremental improvement of the objective function becomes insignificant. Convergence to a minimum is ensured by the fact that each step reduces the objective function.

The computation effort for a full search over all solutions of (17) and (18) grows exponentially with the number of channels. As mentioned above, an alternative is to search only over a linearly independent subset of the solutions. For $\ell_k = 0$, Fig. 2 shows the value of the objective function $C(\alpha)$ versus the computational cost for a full and a reduced search. Herein, the cost is measured in terms of the number of eigenvalue problems (5) that have to be solved. We see that with respect to the computation effort, a search over the subset $\mathcal{B}'$ converges more rapidly than the full search over $\mathcal{B}$. The final results are equal in both cases. Thus, for large M, only a reduced search should be performed. A second view of the convergence properties is provided in Fig. 3. For the initialization of the filter design process, the "lazy prototype" and the ELT prototype [17], respectively, were used. The lazy prototype, which only has M subsequent coefficient being equal to one, simply provides the polyphase transform of the input signal. As we see in Fig. 3, the design process rapidly converges from both initializations. The difference in the convergence behavior for $M = 8$ and $M = 16$ is explained by the fact that for $M = 16$, only a reduced search was performed.

In order to show how longer filters can be designed from shorter ones, the filter length of the eight-band prototype in Fig. 3 was increased every 142 iterations by applying a step with $\ell_k = 1$ for all polyphase components. As we see, the objective function decreases with increasing filter length, but for $L > 6M$, the improvement becomes insignificant. However, for short filters, the convergence properties are excellent.

From the theoretical point of view, a proof of completeness (in the sense that every possible solution can be reached via the proposed searches) would be desirable. However, even if completeness could be shown, it may not be expected that the global optimum can be reached

$$\begin{bmatrix} -A_M(z), & A_{M+1}(z), & A_{M+2}(z), & \cdots \\ z^{-(m-1)}\tilde{A}_M(z), & A_{M+1}(z), & A_{M+2}(z), & \cdots \\ A_M(z), & -A_{M+1}(z), & A_{M+2}(z), & \cdots \\ A_M(z), & z^{-(m-1)}\tilde{A}_{M+1}(z), & A_{M+2}(z), & \cdots \\ A_M(z), & A_{M+1}(z), & -A_{M+2}(z), & \cdots \\ A_M(z), & A_{M+1}(z), & z^{-(m-1)}\tilde{A}_{M+2}(z), & \cdots \\ \vdots & \vdots & & \end{bmatrix} \tag{19}$$

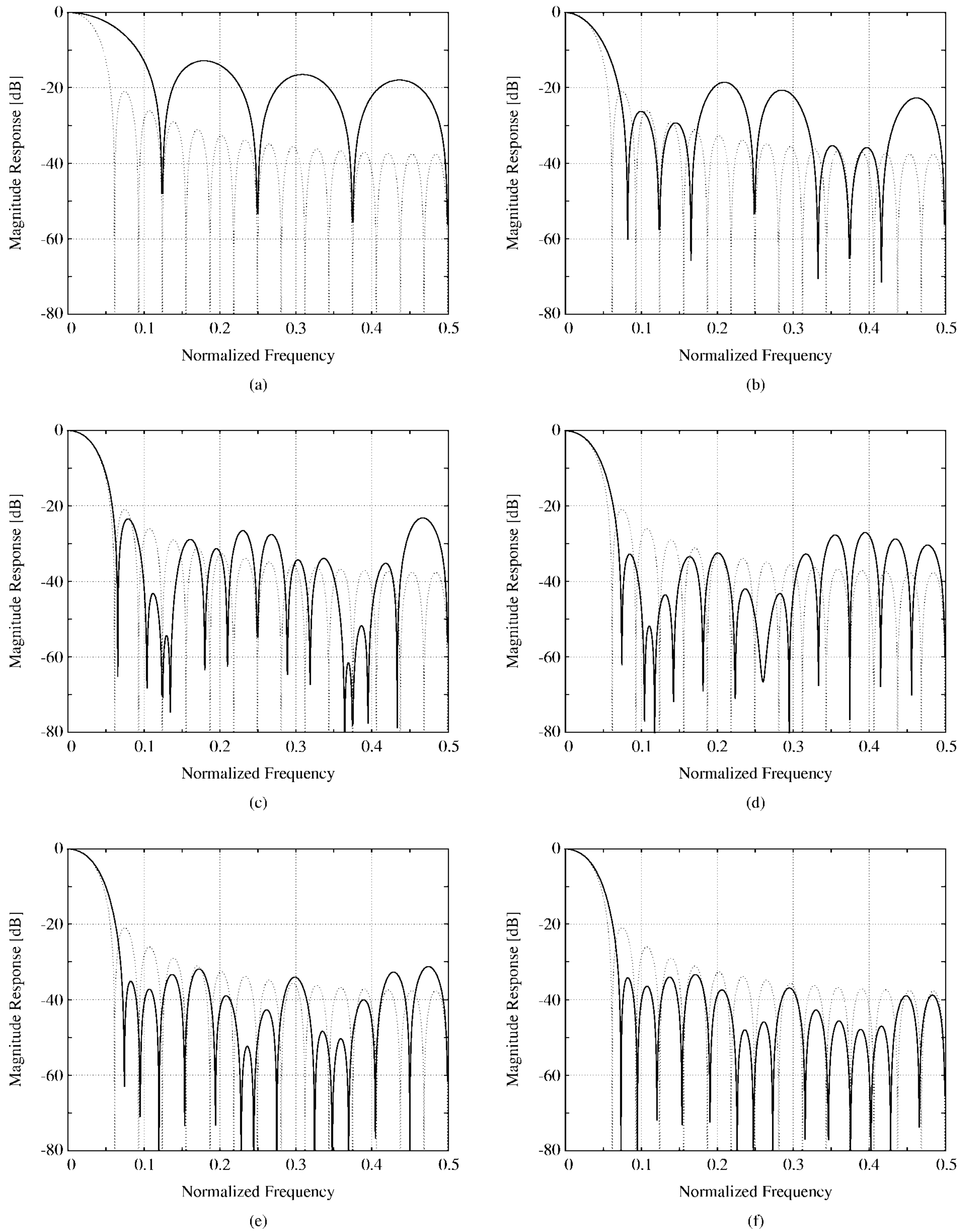

Fig. 6. Frequency responses for the eight-channel prototypes from Table II. For comparison purposes, the frequency response of MALVAR'S ELT prototype [17] is depicted with dotted lines.

via a series of optimal intermediate steps. As with other methods, such as lattice factorizations, the whole series of design steps would have to be considered as one parameterization, and the optimization would have to be based on nonlinear optimization routines.

Design Examples: We consider an eight-channel filter bank, where the filter length is chosen as $L = 32$. The ELT prototype [17] was taken as an initial solution, and the filter was then optimized according to (2) for the stopband edge at $\omega_s = \pi/M$ and a constant weight within the stopband. Fig. 4 shows the magnitude frequency responses of the initial and the optimized filter. The same optimal filter is found with the method from [12]. For longer filters ($L > 4M$), the results for the method from [12] were better than for the proposed algorithm;

see Table I. This means that the main advantage of the proposed technique lies in the fact that integer-valued PR prototypes can be designed (see the next section).

Relation to Lifting Schemes: Somehow, the design procedure proposed in this correspondence has similarities with the lifting scheme, which was originally proposed for the successive design of wavelets from the polyphase transform [18]. However, the essential difference between both methods is that the subspace approach considers linear combinations of impulse responses, whereas the lifting scheme considers factorizations. Although lifting is complete in the sense that every prototype for a cosine-modulated filter bank can be constructed via lifting [10], the way of combining given polyphase components to new ones is completely different. For example, a linear combination of the form $A_k^{(new)}(z) = \alpha_1 A_k(z) + \alpha_2 \tilde{A}_{M+k}(z)$, $A_{M+k}^{(new)}(z) = \alpha_1 A_{M+k}(z) - \alpha_2 \tilde{A}_k(z)$ is not easily realized with a few lifting steps.

V. Prototypes with Discrete Coefficients

One remarkable feature of the relations (17) and (18) is the fact that the prototype $B(z)$ essentially has the same coefficients as the prototype $A(z)$; the filter $B(z)$ is composed from flipped and/or sign-changed polyphase components of the filter $A(z)$. This means that if we start the design with a PR filter $A(z)$ having only integer coefficients, and if we also use integer weights α_1 and α_2 in (9), then the linear combination of $A(z)$ and $B(z)$ will have integer coefficients. The simplest choice for the initial filter with integer coefficients is the lazy filter. In order to design PR filters with integer coefficients, we still need to quantize the weights: $\alpha'_k := \text{round}(\nu\, \alpha_k)$, $k = 1, 2$, $\nu \in \mathbb{R}$.

Filter optimizations have been carried out for $M = 4, 8$, and 16 bands while starting from the lazy filter. The filter lengths were restricted to be $L = 4M$. Various prototype coefficients with different wordlengths are listed in Tables II–IV. Because of symmetry, only the first $2M$ coefficients are listed.

In order to illustrate the design steps and the influence of the wordlengths, the impulse responses for the eight-band case from Table III are depicted in Fig. 5, and the corresponding frequency responses are shown in Fig. 6. For example, the filter (c) with coefficients in the range $[-1, 8]$ has an acceptable frequency response for image coding purposes, while the implementation cost is extremely low.

Other Discrete Implementation: As mentioned before, prototypes can be realized using lattices that allow quantization of the rotation angles without loss of the PR property [11], [17]. However, since finite-wordlength implementations of rotations require the quantization of sets like $(\sin\theta, \cos\theta)$ or $(\cos\theta, \tan\theta)$, rather than the quantization of the rotation angle θ itself, the lattice realizations are restricted to certain discrete-realizable rotation angles if PR is desired [15], [19]. The major problem that arises with such discrete implementations is in achieving equal scaling for all polyphase filters. Overall, the answer to the question of which implementation performs best in practice highly depends on the prototype, the coefficient precision, and the type of hardware realization for which the filter is designed (general-purpose DSP, VLSI with shift and add architectures, etc.).

In addition to the discrete implementation of the polyphase filters, a discrete realization of the modulation part of the filter bank may also be desirable. This can be done by replacing the rotations being involved in the computation of the cosine-modulation through μ rotations, as proposed in [19] for the DCT-II.

VI. Conclusions

In this paper, a novel method for the design of prototypes for PR cosine-modulated filter banks has been presented. The approach is iterative, and the PR property is preserved throughout the optimization. Each iteration step consists of the computation of optimal linear combinations of impulse responses. The linear combinations have to be performed for filters from suitable linear subspaces. The computational cost of the filter optimization is extremely low. The most important feature of the new design method is the fact that it allows the design of PR prototypes with integer-valued coefficients, which are desirable for efficient hardware realizations.

Acknowledgment

The author wishes to thank T. Karp for her helpful discussions and the anonymous reviewers for their suggestions, which improved the manuscript.

References

[1] J. P. Princen and A. B. Bradley, "Analysis/synthesis filter bank design based on time domain aliasing cancellation," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-34, pp. 1153–1161, Oct. 1986.

[2] J. H. Rothweiler, "Polyphase quadrature filters—A new subband coding technique," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.,* 1983, pp. 1280–1283.

[3] H. S. Malvar, "Extended lapped transforms: Fast algorithms and applications," *IEEE Trans. Signal Processing,* vol. 40, Nov. 1992.

[4] T. A. Ramstad and J. P. Tanem, "Cosine modulated analysis-synthesis filter bank with critical sampling and perfect reconstruction," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.,* Toronto, Ont., Canada, May 1991, pp. 1789–1792.

[5] R. D. Koilpillai and P. P. Vaidyanathan, "Cosine-modulated FIR filter banks satisfying perfect reconstruction," *IEEE Trans. Signal Processing,* vol. 40, pp. 770–783, Apr. 1992.

[6] J. Mau, "Perfect reconstruction modulated filter banks, fast algorithms and attractive new properties," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.,* Minneapolis, MN, Apr. 1993, pp. III.225–III.228.

[7] K. Nayebi, T. P. Barnwell, III, and M. J. T. Smith, "Low delay FIR filter banks: Design and evaluation," *IEEE Trans. Signal Processing,* vol. 42, pp. 24–31, Jan. 1994.

[8] G. Schuller and M. J. T. Smith, "A new framework for modulated perfect reconstruction filter banks," *IEEE Trans. Signal Processing,* vol. 44, pp. 1941–1954, Aug. 1996.

[9] P. Heller, T. Karp, and T. Nguyen, "A formulation of modulated filter banks," submitted for publication.

[10] T. Karp and A. Mertins, "Lifting schemes for biorthogonal modulated filter banks," in *Proc. Int. Conf. Digital Signal Process.,* Santorini, Greece, June 1997, vol. 1, pp. 443–446.

[11] P. P. Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[12] T. Q. Nguyen, "Digital filter bank design quadratic constraint formulation," *IEEE Trans. Signal Processing,* vol. 43, pp. 2103–2108, Sept. 1995.

[13] H. Xu, W.-S. Lu, and A. Antoniou, "Efficient iterative design method for cosine-modulated QMF banks," *IEEE Trans. Signal Processing,* vol. 44, pp. 1657–1668, July 1996.

[14] C. Creusere and S. Mitra, "A simple method for designing high-quality prototype filters for m-band pseudo QMF banks," *IEEE Trans. Signal Processing,* vol. 43, pp. 1005–1007, Apr. 1995.

[15] T. Karp, A. Mertins, and T. Q. Nguyen, "Efficiently VLSI-realizable prototype filters for modulated filter banks," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.,* Munich, Germany, Apr. 1997, vol. 3, pp. 2445–2448.

[16] H. S. Malvar, "Lapped transforms for efficient transform/subband coding," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. 38, pp. 969–978, June 1990.

[17] H. S. Malvar, *Signal Processing with Lapped Transforms.* Norwood, MA: Artech House, 1992.

[18] W. Sweldens, "The lifting scheme: A custom design construction of biorthogonal wavelets," *J. Appl. Comput. Harmonic Anal.,* vol. 3, no. 2, pp. 186–200, 1996.

[19] P. Rieder, J. Götze, M. Sauer, and J. A. Nossek, "Orthogonal approximation of the discrete cosine transform," in *Proc. Euro. Conf. Circuit Theory Des., ECCTD,* Istanbul, Turkey, Aug. 1995, pp. 1003–1006.