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INTEGER–MODULATED FILTER BANKS PROVIDING PERFECT RECONSTRUCTION

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ABSTRACT

In this paper, we extend the perfect reconstruction conditions known for cosine modulation to other, more general modulation schemes. The modified PR conditions provide additional degrees of freedom which can be utilized to design integer-modulated filter banks. Techniques for the design of prototypes and modulation sequences are presented.

1 INTRODUCTION

When a perfect reconstruction (PR) filter bank is to be implemented on a processor with finite-precision arithmetic, the prototype and the modulating sequences usually have to be quantized and the PR property gets lost. It is therefore of significant interest to have filter banks that allow PR also with finite precision arithmetic. Integer-coefficient prototypes were designed in [1, 2]. Design methods for PR filter banks with integer modulation and integer prototypes have been presented in [3, 4]. In [3] the integer modulation sequences are designed on the basis of the dyadic symmetry principle [5], and the filter lengths, L, are restricted to the case $L = 2M$, where M is the number of bands. In [4] prototypes with lengths $L \geq 2M$ are considered, and design methods for modulation sequences which are not restricted to dyadic symmetry are presented.

In this paper, we generalize the methods from [4] by giving a new formulation of the PR conditions, which trades off conditions on the prototype to the modulation sequences (and vice versa) and results in greater design freedom for the overall system.

The paper is organized as follows: First, we establish a set of PR conditions which essentially resemble the properties of cosine modulation, but give the desired degrees of freedom that allow for the design of integer modulated filter banks. Then we outline methods for the prototype and modulation–sequence design resulting in PR. Finally, we present examples of PR integer–modulated filter banks.

2 THE PR CONDITIONS FOR INTEGER MODULATION

We consider critical subsampling, an even number of bands (M), use of the same FIR prototype $p(n)$ for both analysis and synthesis, and an overall delay of $D = 2sM + 2M - 1$ with s being an integer. Note that these are the most common choices in the design of cosine-modulated filter banks. The analysis and synthesis filters, denoted as $h_k(n)$ and $g_k(n)$, $k = 0, \ldots, M-1$ are derived as

$$h_k(n) = p(n)\, t_{1,k}(n), \quad g_k(n) = p(n)\, t_{2,k}(n), \tag{1}$$

for $n = 0, 1, \ldots, L-1$, where the sequences $t_{1,k}(n)$ and $t_{2,k}(n)$ provide the modulation. Independent of the type of modulation, the analysis and synthesis polyphase matrices can be written as

$$\boldsymbol{E}(z) = \boldsymbol{T}_1 \begin{bmatrix} \boldsymbol{P}_0(z^2) \\ z^{-1}\boldsymbol{P}_1(z^2) \end{bmatrix}, \tag{2}$$

$$\boldsymbol{R}(z) = \begin{bmatrix} z^{-1}\boldsymbol{Q}_1(z^2), & \boldsymbol{Q}_0(z^2) \end{bmatrix} \boldsymbol{T}_2^T, \tag{3}$$

with $[\boldsymbol{T}_1]_{k,n} = t_{1,k}(n)$, $[\boldsymbol{T}_2]_{k,n} = t_{2,k}(n)$, $k = 0, 1, \ldots, M-1$, $n = 0, 1, \ldots, 2M-1$. The matrices $\boldsymbol{P}_0(z^2)$, $\boldsymbol{P}_1(z^2)$, $\boldsymbol{Q}_0(z^2)$, $\boldsymbol{Q}_1(z^2)$ are defined as

$$\boldsymbol{P}_0(z^2) = \text{diag}\left[P_0(-z^2), \ldots, P_{M-1}(-z^2)\right], \tag{4}$$

$$\boldsymbol{Q}_0(z^2) = \text{diag}\left[P_{M-1}(-z^2), \ldots, P_0(-z^2)\right], \tag{5}$$

$$\boldsymbol{P}_1(z^2) = \text{diag}\left[P_M(-z^2), \ldots, P_{2M-1}(-z^2)\right], \tag{6}$$

$$\boldsymbol{Q}_1(z^2) = \text{diag}\left[P_{2M-1}(-z^2), \ldots, P_M(-z^2)\right] \tag{7}$$

with $P_j(z) = \sum_\ell p(2\ell M + j)\, z^{-\ell}, \quad j = 0, \ldots, 2M-1$. The PR conditions can be formulated in the polyphase domain as

$$\boldsymbol{R}(z)\boldsymbol{E}(z) = z^{-2s-1}\, \gamma\, \boldsymbol{I}_M, \tag{8}$$

where $\boldsymbol{I}_M$ is the $M \times M$ identity matrix. The factor γ in (8) is introduced, because (without additional scaling) integer arithmetic will not allow us to have an overall amplification of one.

For the commonly used cosine modulation

$$t_{1,k}(n) = 2\cos\left[\frac{\pi}{M}\left(k + \frac{1}{2}\right)\left(n - \frac{D}{2}\right) + \phi_k\right], \tag{9}$$

$$t_{2,k}(n) = 2\cos\left[\frac{\pi}{M}\left(k + \frac{1}{2}\right)\left(2M - 1 - n - \frac{D}{2}\right) - \phi_k\right] \tag{10}$$

with $\phi_k = (-1)^k \pi/4$, $n = 0, \dots, L-1$, $k = 0, \dots, M-1$, we find that

$$\boldsymbol{T}_2^{\mathrm{T}} \boldsymbol{T}_1 = \varepsilon \begin{bmatrix} (-1)^s \boldsymbol{I}_M + \boldsymbol{J}_M & \boldsymbol{0} \\ \boldsymbol{0} & (-1)^s \boldsymbol{I}_M - \boldsymbol{J}_M \end{bmatrix} \quad (11)$$

with $\varepsilon = 2M$ and $\boldsymbol{J}_M$ being the $M \times M$ counter identity matrix. Equations (2) – (8) and (11) then yield the following PR conditions on the prototype:

$$\left[z^{-1}\boldsymbol{Q}_1(z^2)\,\boldsymbol{Q}_0(z^2)\right] \begin{bmatrix} \boldsymbol{P}_0(z^2) \\ z^{-1}\boldsymbol{P}_1(z^2) \end{bmatrix} = (-1)^s z^{-2s-1} \frac{\gamma}{\varepsilon} \boldsymbol{I}_M \quad (12)$$

After substitution of z for $-z^2$ they simplify to

$$P_{2M-1-k}(z)P_k(z) + P_{M+k}(z)P_{M-1-k}(z) = \frac{\gamma}{\varepsilon} z^{-s} \quad (13)$$

for $k = 0, \dots, \frac{M}{2} - 1$. We see that the conditions (13) on the prototype are separated from the ones in (11) for the modulation scheme. Consequently, any combination of prototypes satisfying (13) and modulation sequences satisfying (11) will result in an PR filter bank.

We now write $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ as

$$\boldsymbol{T}_1 = \boldsymbol{V}_1 \boldsymbol{Y}_1, \quad \boldsymbol{T}_2 = \boldsymbol{V}_2 \boldsymbol{Y}_2 \quad (14)$$

with

$$\boldsymbol{Y}_1 = \begin{bmatrix} (-1)^s \boldsymbol{J}_{M/2} & \boldsymbol{I}_{M/2} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{M/2} & -(-1)^s \boldsymbol{J}_{M/2} \end{bmatrix},$$
$$\boldsymbol{Y}_2 = (-1)^s \boldsymbol{Y}_1.$$

Provided that $\boldsymbol{V}_2^T \boldsymbol{V}_1 = \varepsilon \boldsymbol{I}$ the matrices $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ according to (14) satisfy (11).[1] However, the formulation (14) also allows us to relax the conditions on $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ to

$$\boldsymbol{V}_2^T \boldsymbol{V}_1 = \boldsymbol{\Gamma} \quad (15)$$

with

$$\boldsymbol{\Gamma} = \mathrm{diag}\left[w_{\frac{M}{2}-1}, \dots, w_0, w_0, \dots, w_{\frac{M}{2}-1}\right]. \quad (16)$$

The product $\boldsymbol{T}_2^T \boldsymbol{T}_1$ then takes on the form

$$\boldsymbol{T}_2^T \boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{\Delta}[(-1)^s \boldsymbol{I} + \boldsymbol{J}] & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Delta}[(-1)^s \boldsymbol{I} - \boldsymbol{J}] \end{bmatrix} \quad (17)$$

with

$$\boldsymbol{\Delta} = \mathrm{diag}\left[w_0, \dots, w_{\frac{M}{2}-1}, w_{\frac{M}{2}-1}, \dots, w_0\right]. \quad (18)$$

This finally results in the following PR conditions on the prototype:

$$w_k \left[P_k(z)P_{2M-1-k}(z) + P_{M+k}(z)P_{M-1-k}(z)\right] = \gamma z^{-s} \quad (19)$$

[1] Note that $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ for ideal cosine modulation can also be written in the form (14). $\boldsymbol{V} = \boldsymbol{V}_1 = \boldsymbol{V}_2$ then is a cosine matrix satisfying $\boldsymbol{V}^T \boldsymbol{V} = \varepsilon \boldsymbol{I}$.

Clearly, the conditions (17) and (19) give us increased design freedom compared to (11) and (13), which is an advantage in the design of integer schemes. For example, if an integer matrix $\boldsymbol{V}$ with $\boldsymbol{V}^T \boldsymbol{V} = \varepsilon \boldsymbol{I}$ and integer factors w_k are given, integer matrices $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ can be derived as

$$\boldsymbol{V}_1 = \boldsymbol{V}\boldsymbol{\Gamma}, \quad \boldsymbol{V}_2 = \boldsymbol{V}, \quad (20)$$

or as

$$\boldsymbol{V}_1 = \boldsymbol{V}, \quad \boldsymbol{V}_2 = \boldsymbol{V}\boldsymbol{\Gamma}, \quad (21)$$

with $\boldsymbol{\Gamma}$ according to (16).

3 DESIGN OF FINITE–WORDLENGTH COEFFICIENT PROTOTYPE FILTERS

In this section we discuss two methods for the design of integer–coefficient prototypes for general, including low–delay integer–modulated filter banks. The case of linear–phase integer prototypes has been considered in [2].

3.1 Alteration of Quantized Prototypes

A method to design PR prototypes with integer coefficients is to quantize a given prototype and to alter the obtained coefficients until (19) is satisfied with some factors w_k. Typically, this leads to several possible solutions for the polyphase filters $P_k(z)$, and the best combination can be chosen. Given the factors w_k the matrices $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ can then be designed to satisfy (15).

3.2 Lifting Scheme

For a systematic design of integer-coefficient prototype filters we start with real-valued prototypes satisfying (22) and decompose them into lifting steps using the procedure described below. Integer coefficients are obtained by quantizing the lifting coefficients.

In [6] it has been demonstrated that the PR constraint in (12) can be expressed by $M/2$ separate constraints for $k = 0, \dots, M/2 - 1$:

$$\boldsymbol{P}_{s,k}(z) \cdot \boldsymbol{P}_{a,k}(z) = \frac{\gamma}{\varepsilon} (-1)^s z^{-2s-1} \boldsymbol{I}_2 \quad (22)$$

with

$$\boldsymbol{P}_{a,k}(z) = \quad (23)$$
$$\begin{bmatrix} P_k(-z^2) & (-1)^s P_{M-1-k}(-z^2) \\ (-1)^{s-1} z^{-1} P_{k+M}(-z^2) & z^{-1} P_{2M-1-k}(-z^2) \end{bmatrix}$$

$$\boldsymbol{P}_{s,k}(z) = \quad (24)$$
$$\begin{bmatrix} z^{-1} P_{2M-1-k}(-z^2) & (-1)^{s-1} P_{M-1-k}(-z^2) \\ (-1)^s z^{-1} P_{k+M}(-z^2) & P_k(-z^2) \end{bmatrix}$$

Each matrix can furthermore be decomposed in the following way:

$$\boldsymbol{P}_{a,k}(z) = \prod_{j=1}^{j_0} \boldsymbol{D}_{k,j}(z) \prod_{i=1}^{i_0} \boldsymbol{B}_{k,i}(z) \cdot \boldsymbol{P}_{a,k,ini}(z) \quad (25)$$

$$\boldsymbol{P}_{s,k}(z) = \boldsymbol{P}_{s,k,ini}(z) \prod_{i=i_0}^{1} \boldsymbol{B}_{k,i}^{-1}(z) \prod_{j=j_0}^{1} \left(z^{-\delta_{k,j}-1} \boldsymbol{D}_{k,j}^{-1}(z)\right) \quad (26)$$

where j_0 can be computed from $\sum_{j=1}^{j_0}(\delta_{\ell,j}+1) = 2s$ for a fixed value of s in (22). The matrices $\boldsymbol{B}_{k,i}$ in (25) are called Zero-Delay matrices since they basically result in longer filters with the same system delay while matrices $\boldsymbol{D}_{k,j}(z)$ are called Maximum-Delay matrices because they enlarge the system delay, and $\boldsymbol{P}_{a,k}(z)$ and $\boldsymbol{P}_{s,k}(z)$ are initialization matrices defined as follows:

$$\boldsymbol{B}_{k,i}(z) = \begin{bmatrix} 0 & 1 \\ 1 & b_{k,i}\, z^{-\beta_{k,i}} \end{bmatrix},\ \boldsymbol{B}_{k,i}^{-1}(z) = \begin{bmatrix} -b_{k,i}\, z^{-\beta_{k,i}} & 1 \\ 1 & 0 \end{bmatrix} \tag{27}$$

$$\boldsymbol{D}_{k,j}(z) = \begin{bmatrix} d_{k,j} & z^{-1} \\ z^{-\delta_{k,j}} & 0 \end{bmatrix},\ z^{-\delta_{k,j}-1}\boldsymbol{D}_{k,j}^{-1}(z) = \begin{bmatrix} 0 & z^{-1} \\ z^{-\delta_{k,j}} & -d_{k,j} \end{bmatrix} \tag{28}$$

$$\boldsymbol{P}_{a,k,ini}(z) = (-1)^s \begin{bmatrix} 1 & 0 \\ p_{k,1}\, z^{-1} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & p_{k,2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p_{k,3} & 1 \end{bmatrix} \tag{29}$$

$$\boldsymbol{P}_{s,k,ini}(z) = \frac{\gamma}{\epsilon} \begin{bmatrix} 1 & 0 \\ -p_{k,3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -p_{k,2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ -p_{k,1}\, z^{-1} & 1 \end{bmatrix} \tag{30}$$

The variables $b_{k,i},\ d_{k,j},\ p_{k,0},\ p_{k,1},\ p_{k,2},\ p_{k,3}$ are real valued coefficients and $\beta_{k,i}, \delta_{k,j}$ are non-negative integer values. Since all matrices and their inverses contain the same coefficients, coefficient quantization does not affect the PR property of the filter bank. However, the frequency response of the prototype is affected by quantization.

To obtain a set of quantized coefficients we apply the following algorithm:

1. In a first step we perform a fine quantization. Each coefficient is expressed using signed binary number representation with a given word length w and a maximum number of ± 1 entries. This quantization should be such that it does not affect the frequency response of the original, unquantized prototype filter in a significant way.
2. We then use a pruning algorithm to obtain a coarser quantization. Iteratively we calculate for all coefficients a signed binary representation with one ± 1 entry less and determine the increase of the stopband energy of the prototype filter that is caused by this step. Then the coarser quantization is only applied to the coefficient resulting in the least increase of the stopband energy.
3. The whole procedure is repeated as long as the designer decides that the frequency response is still tolerable.

4 INTEGER MODULATION MATRICES

Two techniques for the design of suitable integer modulation matrices have been discussed in [4]. The first one is based on an orthogonal projection technique and is suitable for the design of matrices containing small integers. The second one is based on Householder factorizations and typically results in modulation sequences containing larger integers. The reader is referred to [4] for more details. In the following we outline a trial and error method which preserves the symmetries found in the ideal cosine sequences. Note that for $M = 8$ and $D = 15$ the ideal cosine matrix $\boldsymbol{V}$ takes on the form

$$\boldsymbol{V} = \begin{bmatrix} a & b & c & d & e & f & g & h \\ -b & -e & -h & f & c & a & d & g \\ -c & -h & d & b & g & -e & -a & -f \\ d & -f & -b & h & a & g & -c & -e \\ e & -c & -g & a & -h & -b & f & d \\ -f & a & -e & -g & b & -d & -h & c \\ -g & d & -a & c & -f & -h & e & -b \\ h & -g & f & -e & d & -c & b & -a \end{bmatrix} \tag{31}$$

Similar structures occur for other M. Through variation of the M free parameters (in the above case: $a, b, \ldots, h$) suitable matrices $\boldsymbol{V}$ can be found.

Table 1: parameters for integer prototype.

p_{01}	$-1+2^{-2}$	p_{21}	$-1+2^{-3}$
p_{02}	$1-2^{-4}$	p_{22}	$2^{-1}+2^{-2}$
p_{03}	$-1+2^{-4}$	p_{23}	$-1+2^{-4}$
b_{01}	$2^{-2}+2^{-4}$	b_{21}	$2^{-1}+2^{-4}$
b_{02}	2^{-1}	b_{22}	$2^{-2}-2^{-4}$
p_{11}	$-1+2^{-2}$	p_{31}	$-1-2^{-3}$
p_{12}	$1-2^{-2}$	p_{32}	$2^{-1}+2^{-4}$
p_{13}	$-1+2^{-4}$	p_{33}	$-1+2^{-4}$
b_{11}	2^{-1}	b_{31}	1
b_{12}	$2^{-2}+2^{-4}$	b_{32}	2^{-4}

5 DESIGN EXAMPLES

In a first example we consider a modulated filter bank with $M = 8$ channels, a filter length of $L = 32$, and an overall system delay of the filter bank of $D = 15$ samples. The prototype is designed with the lifting scheme in Section 3.2. In the quantization step the word length is set to 8 bits and the number of ± 1 entries to 3 for the initialization matrix and to 2 for the low-delay matrices. Note that in this example the delay is minimal such that we do not have any maximum delay matrices in the factorization. After several steps of pruning we obtain the values given in Table 1, and $\beta_{k,i} = 1$ for $k = 0, 1, 2, 3$ and $i = 1, 2$. The frequency responses of the resulting analysis filters are shown in Figure 1.

For the modulation we use the matrix $\boldsymbol{V}$ in (32) obtained by the method outlined in Section 4.

$$\boldsymbol{V} = \begin{bmatrix} 27 & 28 & 24 & 23 & 19 & 14 & 9 & 5 \\ -28 & -19 & -5 & 14 & 24 & 27 & 23 & 9 \\ -24 & -5 & 23 & 28 & 9 & -19 & -27 & -14 \\ 23 & -14 & -28 & 5 & 27 & 9 & -24 & -19 \\ 19 & -24 & -9 & 27 & -5 & -28 & 14 & 23 \\ -14 & 27 & -19 & -9 & 28 & -23 & -5 & 24 \\ -9 & 23 & -27 & 24 & -14 & -5 & 19 & -28 \\ 5 & -9 & 14 & -19 & 23 & -24 & 28 & -27 \end{bmatrix} \tag{32}$$

In a second design example we maintain the number of subbands and the filter length but consider a linear phase prototype, resulting in a paraunitary integer-modulated filter

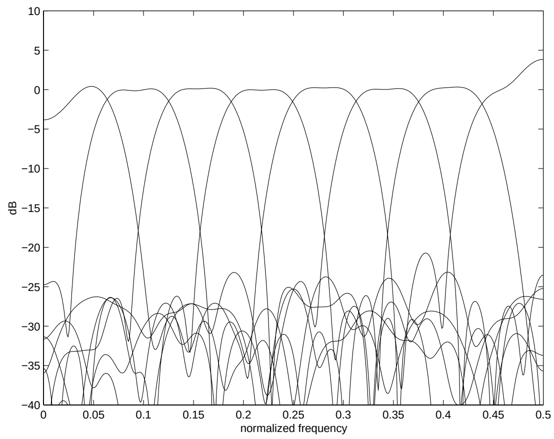

Figure 1: Normalized frequency response of low-delay integer-modulated filter bank with $M = 8$ subbands, $D = 15$, filter length $L = 32$, and coefficients from Table 1.

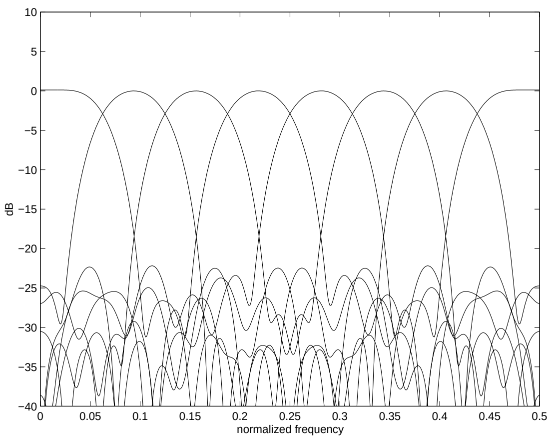

Figure 2: Normalized frequency response of integer-modulated paraunitary filter bank with $M = 8$ subbands and filter length $L = 32$.

bank (up to a scale factor). The prototype is given by [2]

$$\begin{aligned} h(n) = \{ & -6,\ -4,\ 0,\ -6,\ 7,\ 0,\ 8,\ 17,\ 24,\ 33,\ 41, \\ & 48,\ 56,\ 62,\ 66,\ 68,\ 68,\ 66,\ 62,\ 56,\ 48, \\ & 41,\ 33,\ 24,\ 17,\ 8,\ 0,\ 7,\ -6,\ 0,\ -4,\ -6\} \end{aligned}$$

and the modulation matrix $\boldsymbol{V}$ is identical to the one in the last example. Figure 2 depicts the analysis filters' frequency responses.

A third example presents results for the design method described in Section 3.1. Again, we consider $M = 8$ bands and $D = 15$, but the filter length is now 44. The prototype is given by

$$\begin{aligned} h(n) = \{ & 2,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 16,\ 18,\ 16,\ 17,\ 16,\ 15, \\ & 13,\ 12,\ 9,\ 6,\ 5,\ 3,\ 0,\ -2,\ -2,\ -3,\ -4,\ -3,\ -3,\ -3, \\ & -3,\ -2,\ -1,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 1\}. \end{aligned}$$

The prefactors are $w_0 = 273 \times 251 \times 265$, $w_1 = 264 \times 251 \times 265$, $w_2 = 264 \times 273 \times 265$, $w_3 = 264 \times 273 \times 251$. The modulation matrices are constructed according to (21) with $\boldsymbol{V}$ as in

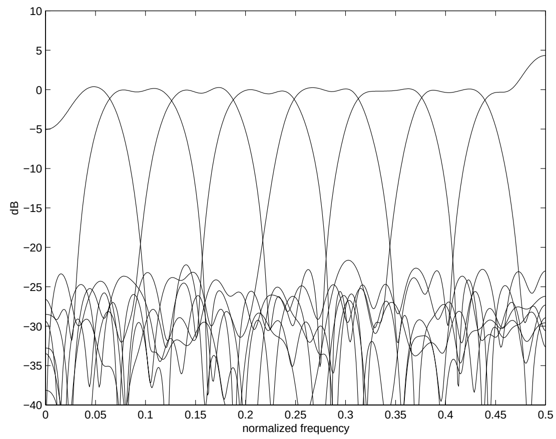

Figure 3: Normalized frequency response of low-delay integer-modulated filter bank with $M = 8$ subbands, $D = 15$, and filter length $L = 44$.

(32). The frequency responses of the resulting analysis filters are depicted in Figure 3.

6 CONCLUSIONS

We have shown that it is possible to design integer-modulated filter banks with perfect reconstruction and good frequency selectivity. Future work will be directed towards finding filter banks with a large number of bands and efficient integer implementations.

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