STATISTICAL OPTIMIZATION OF PR-QMF BANKS AND WAVELETS

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ABSTRACT

The design of optimal power complementary filters for multiresolution signal decomposition is considered. As a new performance measure in statistical optimization of filter banks the norm of cross-correlation (all subbands including all delays) will be used for optimization. The obtained filters will be compared to filters having minimum reconstruction error when the high-pass branches are dropped. In the paraunitary case, which is considered here, this reconstruction error is closely related to the coding gain of PCM in subbands over direct PCM. While the KL-transform combines both minimum reconstruction error and perfect decorrelation, filter banks do not. It is clear that the norm of cross-correlation becomes very important when better techniques than PCM are considered for the coding of subband signals.

1 INTRODUCTION

Recently, multiresolution signal decomposition has been successfully applied to image and audio coding. The main idea of subband decomposition is splitting the input signal into uncorrelated subband signals (at lower rates) which can be processed like independent signals. In practice, filters with overlapping frequency bands must be used to provide a perfect reconstruction (PR) analysis/synthesis system. Therefore perfect decorrelation cannot be achieved.

In statistical optimization of PR-QMF systems a prototype filter $H(z)$ which extremizes an optimality criterion has to be found. Several researchers have focused on maximizing the coding gain of PCM in subbands or minimizing the reconstruction error when the high-pass branches are dropped [1, 3, 5, 6, 7, 10]. These optimality criteria will be briefly reviewed in sections 2 and 3. If other coding techniques like VQ [2] are applied in the subbands, the decorrelation property of the filter bank clearly becomes more important than the energy compaction property.

Paraunitary filter banks are considered throughout this paper. This means that a prototype filter $H(z)$ which satisfies the power complementary property

$$|H(e^{j\Omega})|^2 + |H(e^{j(\pi+\Omega)})|^2 = 2 \quad \forall\, \Omega \qquad (1)$$

has to be found.

The analysis and synthesis filters (see figure 1) are related to the prototype $H(z)$ as follows:

$$\begin{aligned} H_0(z) &= H(z) \\ H_1(z) &= z^{-(L-1)}\, H(-z^{-1}) \\ G_0(z) &= H_0(z^{-1}) \\ G_1(z) &= H_1(z^{-1}) \end{aligned} \qquad (2)$$

Usually the subband signals are computed successively from a given input sequence $c_N(n) = x(n)$, but for the formulation of optimality criteria it is useful to write the subband signals in the form (see figure 2)

$$\begin{aligned} c_{N-k}(m) &= \textstyle\sum_{\ell=0}^{L_k-1} c_N(\ell)\, a_k(2^k m - \ell), \\ d_{N-k}(m) &= \textstyle\sum_{\ell=0}^{L_k-1} c_N(\ell)\, b_k(2^k m - \ell) \end{aligned} \qquad (3)$$

where the filters $A_k(z)$ and $B_k(z)$ and the filter lengths L_k are given by

$$\begin{aligned} A_k(z) &= \textstyle\prod_{i=0}^{k-1} H_0(z^{2^i}), \\ B_k(z) &= \begin{cases} H_1(z), & k=1, \\ H_1(z^{2^{k-1}}) \prod_{i=0}^{k-2} H_0(z^{2^i}), & k>1, \end{cases} \\ L_{k+1} &= 2L_k + L - 2, \quad L_1 = L. \end{aligned} \qquad (4)$$

L is the length of the prototype filter.

Note that the subband signals $d_{N-k}(m)$ can be interpreted as coefficients of an orthonormal discrete wavelet transform (DWT) if the filter prototype $H(z)$ satisfies the regularity condition [4].

2 ZONAL SAMPLING

The approximation problem shown in figure 3 appears when one or more high-pass branches are dropped (zonal sampling).

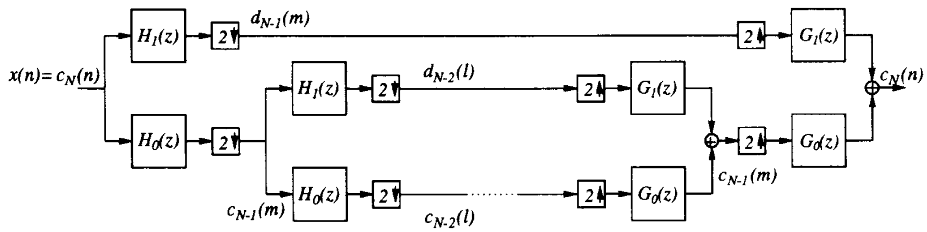

Figure 1: Filter bank

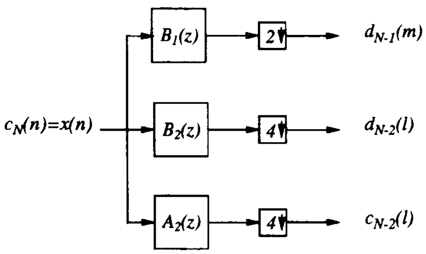

Figure 2: Alternative analysis filter bank

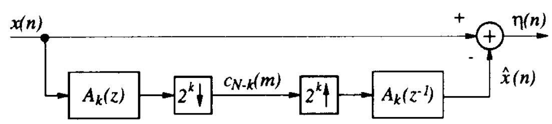

Figure 3: Approximation

Due to the orthonormality of the basis vectors (synthesis impulse responses) the error signal $\eta(n) = x(n) - \hat{x}(n)$ is orthogonal to the output signal $\hat{x}(n)$. Hence, for the variances

$$\sigma_x^2 = E\{|x(n)|^2\}\,,\ \sigma_\eta^2 = E\{|\eta(n)|^2\}\,,\ \sigma_{\hat{x}}^2 = E\{|\hat{x}(n)|^2\}$$

follows

$$\sigma_\eta^2 = \sigma_x^2 - \sigma_{\hat{x}}^2. \tag{5}$$

From the Parseval indentity we have

$$\sigma_{\hat{x}}^2 = 2^{-k}\,\sigma_{c_{N-k}}^2. \tag{6}$$

Formulae (5) and (6) state that the variance $\sigma_{c_{N-k}}^2$ should be maximized when the high-pass branches are dropped. Therefore, the optimality criterion is

$$\sigma_{c_{N-k}}^2 \stackrel{!}{=} \max \tag{7}$$

The variance $\sigma_{c_{N-k}}^2$ itself is given by

$$\sigma_{c_{N-k}}^2 = \boldsymbol{a}_k^T \boldsymbol{R}_{xx}^{(k)} \boldsymbol{a}_k \tag{8}$$

where $\boldsymbol{R}_{xx}^{(k)}$ is the $L_k \times L_k$ autocorrelation matrix of the stationary input process and $\boldsymbol{a}_k$ contains the impulse response of the analysis filter $A_k(z)$.

3 CODING GAIN

The coding gain of PCM in subbands over direct PCM is defined as

$$G_{(\mathrm{SBC})} = \frac{\sigma_{e(\mathrm{PCM})}^2}{\sigma_{e(\mathrm{SBC})}^2} \tag{9}$$

where $e_{(\mathrm{PCM})}(n)$ and $e_{(\mathrm{SBC})}(n)$, respectively, are the reconstruction errors due to quantization at a fixed bit rate. Assuming optimum quantization and uncorrelated quantization errors, the coding gain becomes [6]

$$G_{(\mathrm{SBC})} = \prod_{k=1}^{M} \alpha_k^{-\frac{1}{N_k}} \tag{10}$$

M is the number of subbands and

$$\alpha_k = \begin{cases} \sigma_{d_{N-k}}^2 \,/\, \sigma_x^2, & k = 1, \ldots, M-1, \\ \sigma_{c_{N-(M-1)}}^2 \,/\, \sigma_x^2, & k = M, \end{cases} \tag{11}$$

$$N_k = \begin{cases} 2^k, & k = 1, \ldots, M-1, \\ 2^{M-1}, & k = M. \end{cases} \tag{12}$$

The variance $\sigma_{c_{N-(M-1)}}^2$ is given by (8), and for $\sigma_{d_{N-k}}^2$ we have

$$\sigma_{d_{N-k}}^2 = \boldsymbol{b}_k^T \boldsymbol{R}_{xx}^{(k)} \boldsymbol{b}_k. \tag{13}$$

In the two-band case, the criteria (7) and

$$G_{(\mathrm{SBC})} \stackrel{!}{=} \max \tag{14}$$

are equivalent, and for $M > 2$ they are almost equivalent.

4 CROSS-CORRELATION BASED QUALITY CRITERION

Since the multiresolution decomposition is performed successively, the total amount of cross-correlation between all subbands reaches its minimum if the correlation between the signals $c_{N-k}(m)$ and $d_{N-k}(m)$ is minimized for all k simultaneously. In the literature [1, 3] only $r_{cd}(0) = E\{c(m)d(m)\}$ has been considered. But even if $r_{cd}(0) = 0$ holds, the correlation $r_{cd}(m), m \neq 0$ can take on large values. For measuring cross-correlation (see figures 4 and 5) the norm of all cross-correlation sequences

$$r_{cd}^{(k)}(m) = E\{c_{N-k}(m)d_{N-k}(0)\} \tag{15}$$

will be used:

$$Q(H(z)) = \left[\sum_k \sum_m \left|r_{cd}^{(k)}(m)\right|^2\right]^{\frac{1}{2}} \stackrel{!}{=} \min \tag{16}$$

It can be shown that the cross-correlation is given by

$$r_{cd}^{(k)}(m) = \sum_{i=0}^{L_k-1} \sum_{j=0}^{L_k-1} r_{xx}(j-i+2m)\ a_k(i)\ b_k(j) \tag{17}$$

where $r_{xx}(i-j) = E\{x(i)x(j)\}$ is the autocorrelation of the input signal and L_k is the number of filter taps. Note that the l_p-norm can also be used.

5 PARAMETRIZATION

Parametrizations which allow unconstrained optimization methods are used. These are: (i) Vaidyanathan's and Hoang's lattice structure [9], which leads to general power complementary filters. This means that none of the filters must be a low- or highpass filter; (ii) Zou's and Tewfik's parametrization [8, 5], which leads to filters having a zero at $z = -1$. This condition is necessary if zero mean highpass filters and a wavelet interpretation are desired.

6 RESULTS

The objective functions (7) and (16), respectively, have to be extremized under condition (1), which leads to highly nonlinear relations between the parameters and the objective functions. Therefore a grid search followed by nonlinear optimization has been used.

Frequency responses, scaling functions and cross-correlation $r_{cd}^{(1)}(m)$ of optimal 8-tap filters for AR(1) sources with $\rho = 0.95$ and two-band splitting are shown in figures 4 and 5. The filters are optimized under condition $H(-1) = 0$ and lead to regular scaling functions. The lattice parametrization leads to slightly better results (about 1%), but the scaling functions are not regular.

In what follows, the filter which maximizes coding gain will be denoted by $H_{\text{Gain}}(z)$ and the filter which minimizes correlation will be denoted by $H_{\text{Corr}}(z)$, respectively. The figures show that for optimal decorrelation it is very important that one of the filters has high attenuation for frequencies with high signal energy. These are the low frequencies for an AR(1) source, and it can be seen that the highpass filter $H_{\text{Gain}}(-z^{-1})$ has much less stopband attenuation than the filter $H_{\text{Corr}}(-z^{-1})$. The values of the objective functions are also interesting, see table 1. The energy compaction properties of both filters do not differ very much, but the norm of cross-correlation is better by factor 1.65 for $H_{\text{Corr}}(z)$.

Table 1: Properties of optimal filters

	σ_η^2	$G_{(\text{SBC})}/db$	Q(H(z))
$H_{\text{Gain}}(z)$	0.017117	5.8601	0.016033
$H_{\text{Corr}}(z)$	0.017308	5.8364	0.009679

7 CONCLUSION

As a new performance measure the cross-correlation of subband signals is used for optimization of filter prototypes. Optimal decorrelation is important if the subband signals are processed like independent signals. The results show that filters optimized for decorrelation also lead to good energy compaction while filters optimized for energy compaction do not have good decorrelation properties.

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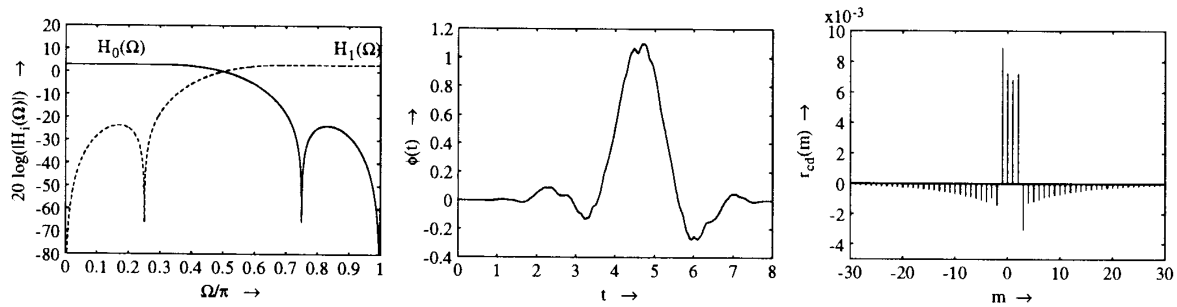

Figure 4: Frequency responses $H_0(e^{j\Omega})$ and $H_1(e^{j\Omega})$, scaling function $\phi(t)$ and cross-correlation $r_{cd}^{(1)}(m)$ for an 8-tap filter with maximum energy compaction.

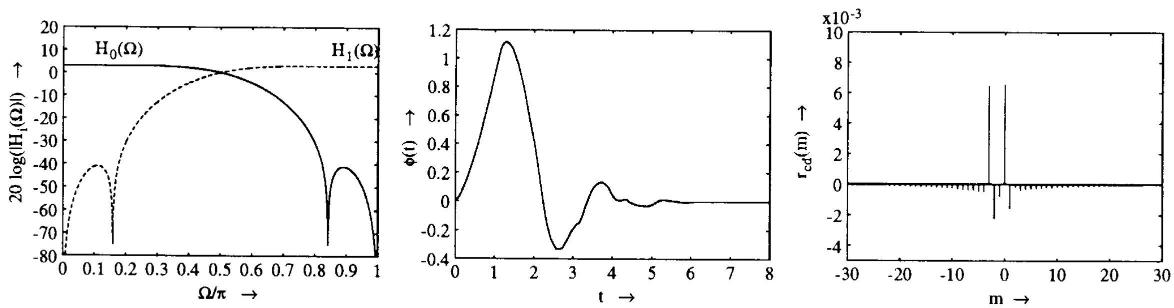

Figure 5: Frequency responses $H_0(e^{j\Omega})$ and $H_1(e^{j\Omega})$, scaling function $\phi(t)$ and cross-correlation $r_{cd}^{(1)}(m)$ for an 8-tap filter with minimum correlation.

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