TIME-VARYING AND SUPPORT PRESERVATIVE FILTER BANKS: DESIGN OF OPTIMAL TRANSITION AND BOUNDARY FILTERS VIA SVD

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ABSTRACT

In this paper, methods for switching filter coefficients and filter bank structures and methods for processing finite length signals will be studied. The problem of designing optimal boundary and transition filters will be solved directly via singular value decomposition (SVD) while the optimality criterion is based on the subband statistics. The optimized filters provide a good match between the subband statistics in transition regions (and at the boundaries) to the statistics in steady state. The filter banks considered are maximally decimated M-channel linear and non-linear phase (biorthogonal and paraunitary) filter banks with real filter coefficients.

1. INTRODUCTION

Several authors have proposed coding schemes based on time-varying filter banks [8], [9]. The idea behind this is to use an optimal filter bank for each region of a signal. This means that we have to switch from one filter bank to another when the signal statistics change. When doing so, we need some transition filters in order to provide an overall perfect reconstruction (PR), or especially, an overall unitary system. However, when we switch from one filter bank to another by using transition filters, we want the characteristics of the transition filters to be as similar as possible to the characteristics of the filters being used in steady state. In other words, the output signals of the transition filters should be almost identical to those for a direct switch of coefficients. This requires optimized transition filters.

When a finite length signal is decomposed by a maximally decimated filter bank, the number of subband samples will be larger than the number of input samples, and simple truncation of the subband sequences will cause distortions at the boundaries. Thus, circular convolution ([1]), signal extensions ([2] - [5]), or special boundary filters ([5] - [10]) are required in order to ensure that N samples can be represented by N coefficients. Furthermore, the subband coding of finite length signals requires that the statistical properties of the subband samples representing the boundaries be equal or almost equal to the properties of the other subband samples (for stationary input signals), because we do not want to change the bit allocations for the last few samples.

In this paper, solutions via SVD to the filter optimization problems will be presented. As initial solutions, only boundary filters derived by symmetric reflection and by the Gram-Schmidt procedure will be considered, because (unlike circular convolution) they lead to local operations.

2. SUPPORT PRESERVATIVE SYSTEMS

In this section, basic results concerning support preservative (SP) systems will be briefly reviewed, and the notation will be introduced. A more detailed description of these algorithms can be found in [5]. For the sake of simplicity, the number of input samples (N) is chosen to be an integer multiple of the number of subbands (M). Furthermore, N is chosen to be large enough to ensure that at least a few samples will be correctly recovered when we analyze the time-limited signal like an infinite length signal using the filter bank in Figure 1, truncate the subband signals to a total number of N non-zero samples and recover the signal like an infinite length signal.

When extension methods are used in order to provide a support preservative system, the analysis section of the SP filter bank can be written as ([5])

$$\boldsymbol{y} = \boldsymbol{\mathcal{H}}^T\, \boldsymbol{x}_E, \qquad \boldsymbol{x}_E = \boldsymbol{E}_x\, \boldsymbol{x}, \qquad \boldsymbol{E}_x = \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{I}_N \\ \boldsymbol{K}_2 \end{bmatrix}. \quad (1)$$

The vector $\boldsymbol{x}$ in (1) contains N input samples, and $\boldsymbol{y}$ contains N subband samples. $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are reflection matrices. The matrix $\boldsymbol{\mathcal{H}}$ has the structure shown in Figure 3. The definition of the $M \times M$ matrices $\boldsymbol{H}(n)$, which are the building blocks of $\boldsymbol{\mathcal{H}}$, is illustrated in Figure 2. As can be seen in Figure 2, the analysis filters are considered as anticausal whereas the synthesis filters are considered as causal (this allows us to avoid an overall delay). Note that all types of reflections reported in literature can be written in the form (1).

Reflection of signals may also be interpreted as reflection of impulse responses which results in a filter bank with special boundary filters. This means that the analysis formula (1) can be written as

$$\boldsymbol{y} = \boldsymbol{F}^T \boldsymbol{x}, \qquad \boldsymbol{F}^T = \boldsymbol{\mathcal{H}}^T \boldsymbol{E}_x, \quad (2)$$

where $\boldsymbol{F}$ is a $N \times N$ matrix.

The most simple approach for finding boundary filters for the paraunitary non-linear phase case is to truncate $\boldsymbol{\mathcal{H}}$ to a $N \times N$ matrix $\boldsymbol{\mathcal{H}}_c$ of rank N,

$$\boldsymbol{\mathcal{H}}^T = [\boldsymbol{\mathcal{H}}_u^T | \boldsymbol{\mathcal{H}}_c^T | \boldsymbol{\mathcal{H}}_l^T], \quad (3)$$

and to orthonormalize it by using the well known Gram-Schmidt procedure ([5], [8]). This results in the $N \times N$ unitary matrix

$$\boldsymbol{F} = [\boldsymbol{F}_l, \boldsymbol{\mathcal{H}}_{cc}, \boldsymbol{F}_r], \quad (4)$$

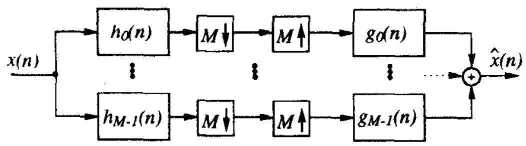


Figure 1: Filter bank in direct form.

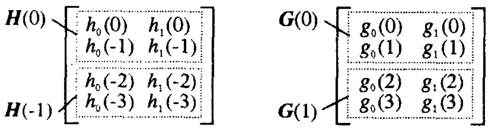


Figure 2: Matrix definitions ($M = 2$, 4-tap filters)

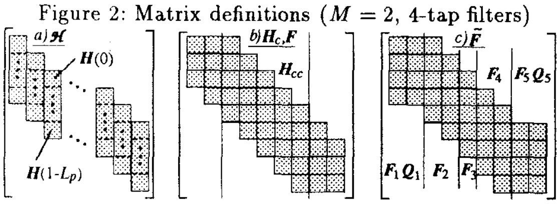


Figure 3: Structures of matrices (for b) and c): $L_p = 5$, $N = 8M$).

where $\boldsymbol{F}_l$ and $\boldsymbol{F}_r$ contain the boundary filters (see Figure 3). The matrix $\mathcal{H}_{cc}$ belongs to the partitioning

$$\mathcal{H}_c = [\mathcal{H}_{cl} \mid \mathcal{H}_{cc} \mid \mathcal{H}_{cr}], \tag{5}$$

where $\mathcal{H}_{cl}$ and $\mathcal{H}_{cr}$ contain the truncated impulse responses.

When paraunitary filter banks are considered, the synthesis equation is simply $\hat{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{y}$. For linear phase filter banks, synthesis is usually performed by extending $\boldsymbol{y}$ symmetrically, filtering, and windowing the output signal. However, the implementation of special boundary filters (derived by reflection or the Gram-Schmidt procedure) will require fewer operations than the implementation of extension methods, and we should implement $\boldsymbol{F}^T$ and $(\boldsymbol{F}^T)^{-1}$ directly.

3. TIME-VARYING FILTER BANKS

As shown in [8], the transition filters for the paraunitary case may be found by using the Gram-Schmidt orthonormalization procedure. However, once we have found two SP systems (matrices $\boldsymbol{F}^{(a)}$ and $\boldsymbol{F}^{(b)}$ according to $\boldsymbol{F}$ in (2) or (4); signal lengths N_a and N_b) we may combine both systems to one SP system for signals of length $N = N_a + N_b$. Clearly, if we do so, we switch from one filter bank to another without overlap. This may cause blocking effects in low bit rate applications, but anyway, it is a posssible strategy and we are not restricted to paraunitary systems. Methods for designing better (overlapping) filters will be presented in the next section.

4. FILTER OPTIMIZATION

In the previous sections, it was outlined how SP filter banks can be achieved and how the switching of filter banks can be performed easily. However, we have not taken the statistics of the subband signals into account. Especially, when we use the Gram-Schmidt procedure, we may end up with boundary filters that lead to undesirable subband statistics. In this section, it will be shown how optimal boundary and transition filters can be found.

In order to get a unified framework for matrices $\boldsymbol{F}$ constructed by reflection or orthonormalization, $\boldsymbol{F}$ and $\boldsymbol{y}$ will be partitioned as

$$\begin{aligned} \boldsymbol{F} &= [\boldsymbol{F}_1|\boldsymbol{F}_2|\boldsymbol{F}_3|\boldsymbol{F}_4|\boldsymbol{F}_5], \\ \boldsymbol{y}^T &= [\boldsymbol{y}_1^T|\boldsymbol{y}_2^T|\boldsymbol{y}_3^T|\boldsymbol{y}_4^T|\boldsymbol{y}_5^T], \quad \boldsymbol{y}_i = \boldsymbol{F}_i^T\boldsymbol{x}. \end{aligned} \tag{6}$$

In (6), the matrices $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ contain the original coefficients of the analysis filters. $\boldsymbol{F}_1$ and $\boldsymbol{F}_5$ contain the boundary filters; $\boldsymbol{F}_2$ has the same size as $\boldsymbol{F}_1$, and $\boldsymbol{F}_4$ has the same size as $\boldsymbol{F}_5$.

4.1 Energies of Subband Signals

Before we start to optimize filters, let us discuss the distribution of signal energy to the boundaries in a single SP system for stationary input processes.

Paraunitary linear-phase filter banks having even-length filters are the most desirable ones, because symmetric reflection methods lead to unitary transform matrices $\boldsymbol{F}$, and due to symmetry and the Parseval identity, we have $E_1 = E_2 = E_4 = E_5$, $E_i = E\{\boldsymbol{y}_i^T\boldsymbol{y}_i\}$. This is the best precondition for achieving good statistical properties of the subband signals in boundary regions.

Paraunitary non-linear phase filters become important when orthonormal wavelet decompositions are considered. In this case, the energies E_1 and E_5 depend on the prototype filters and may be different to a significant extent. However, once the prototype filters are fixed and the matrix $\mathcal{H}_c$ in (3) is defined, the energies E_1 and E_5 are fixed for given input statistics. For a stationary input process and vectors $\boldsymbol{y}_1$ and $\boldsymbol{y}_5$ of the same length, we have $E_1 + E_5 = 2E_2 = 2E_4$.

When using biorthogonal filter banks, we have more freedom in distributing the subband energies, but we also have to take the energies of the synthesis impulse responses into account. These energies may be different so that quantization errors can be amplified differently.

4.2 Optimization of Boundary Filters for SP Systems

Let us consider a stationary input process, and let the energies of $\boldsymbol{y}_1$ and $\boldsymbol{y}_5$ be fixed. Note that we still have an infinite number of choices for $\boldsymbol{F}_1$ and $\boldsymbol{F}_5$ so that we have the possibility of distributing the energy properly to the coefficients in $\boldsymbol{y}_1$ and $\boldsymbol{y}_5$ by rotating the subspaces spanned by the columns of $\boldsymbol{F}_1$ and $\boldsymbol{F}_5$. This rotation can be expressed by multiplying $\boldsymbol{F}_1$ and $\boldsymbol{F}_5$ with unitary matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_5$:[1]

$$\begin{aligned} \tilde{\boldsymbol{F}} &= [\boldsymbol{F}_1\boldsymbol{Q}_1|\boldsymbol{F}_2|\boldsymbol{F}_3|\boldsymbol{F}_4|\boldsymbol{F}_5\boldsymbol{Q}_5], \\ \tilde{\boldsymbol{y}}^T &= [\tilde{\boldsymbol{y}}_1^T|\boldsymbol{y}_2^T|\boldsymbol{y}_3^T|\boldsymbol{y}_4^T|\tilde{\boldsymbol{y}}_5^T]. \end{aligned} \tag{7}$$

[1] When we rotate the analysis filters, we have to rotate the synthesis filters also. Given an analysis/synthesis system of the form $\boldsymbol{y} = \boldsymbol{F}^T\boldsymbol{x}$, $\hat{\boldsymbol{x}} = (\boldsymbol{F}^T)^{-1}\boldsymbol{y}$, we may express the optimal system as $\tilde{\boldsymbol{y}} = \boldsymbol{B}^T\boldsymbol{F}^T\boldsymbol{x}$, $\hat{\boldsymbol{x}} = (\boldsymbol{F}^T)^{-1}\boldsymbol{B}\tilde{\boldsymbol{y}}$, where $\boldsymbol{B}$ contains all rotation matrices. The structure of $\tilde{\boldsymbol{F}}$ is illustrated in Figure 3.

The following optimization method is based on the idea of linearly predicting the missing samples at the boundaries. To explain this, let $\mathbf{x}$ be a realization of the stochastic process $\boldsymbol{x}$. From (6), we have $\mathbf{y}_2 = \boldsymbol{F}_2^T\mathbf{x}$. Now let us try to compute the vector $\mathbf{y}_2$ once again by shifting the samples in $\mathbf{x}$ upwards by N_E positions and using the matrix $\boldsymbol{Q}_1^T\boldsymbol{F}_1^T$ for analysis (N_E is the number of rows of $\mathcal{H}_u$ in (3) which is equal to the number of samples that would be extended). In matrix notation, the operation can be written as

$$\mathbf{v}_1 = \boldsymbol{Q}_1^T\boldsymbol{F}_1^T\boldsymbol{S}\mathbf{x}, \tag{8}$$

where $\boldsymbol{S}$ has the size $N \times N$ and contains the non-zero elements $\boldsymbol{S}_{i,i+N_E} = 1$, $i = 1, \ldots, N - N_E$. Since we have missing samples in $\boldsymbol{S}\mathbf{x}$, we will not be able to make $\mathbf{v}_1$ equal to $\mathbf{y}_2$ by choosing $\boldsymbol{Q}_1$, but we may view $\boldsymbol{d} = \boldsymbol{y}_2 - \boldsymbol{v}_1$ as a stochastic process and minimize the energy $\phi_1 = E\{\boldsymbol{d}^T\boldsymbol{d}\}$, which can be expressed as the trace of the correlation matrix $E\{\boldsymbol{d}\boldsymbol{d}^T\}$:

$$\phi_1 = \text{trace}\{[\boldsymbol{F}_2 - \boldsymbol{S}^T\boldsymbol{F}_1\boldsymbol{Q}_1]^T\boldsymbol{R}_{xx}[\boldsymbol{F}_2 - \boldsymbol{S}^T\boldsymbol{F}_1\boldsymbol{Q}_1]\}. \tag{9}$$

In a further step, we compute (in mind) a Cholesky decomposition of the autocorrelation matrix of the input process ($\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{R}_{xx}$) and write the objective function as

$$\phi_1 = \left\|\boldsymbol{L}^T\boldsymbol{F}_2 - \boldsymbol{L}^T\boldsymbol{S}^T\boldsymbol{F}_1\boldsymbol{Q}_1\right\|_F^2 \stackrel{!}{=} \min, \tag{10}$$

where norm $\|\cdot\|_F$ is the Frobenius norm. The solution to (9) is given by $\boldsymbol{Q}_1 = \boldsymbol{U}\boldsymbol{V}^T$, where $\boldsymbol{U}\Sigma\boldsymbol{V}^T$ is the SVD of the matrix $\boldsymbol{F}_1^T\boldsymbol{S}\boldsymbol{R}_{xx}\boldsymbol{F}_2$ (see [11] for the subspace rotation problem). Note that since $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ may contain many zero elements, the correct matrix $\boldsymbol{C}$ can still be computed when $\boldsymbol{F}_1, \boldsymbol{S}, \boldsymbol{R}_{xx}$ and $\boldsymbol{F}_2$ are truncated properly. The matrix $\boldsymbol{Q}_5$ that minimizes an objective function ϕ_5 according to (10) can be computed in the same way.

4.3 Optimization of Transition Filters

Let us start with the transition method mentioned in section 3:

$$\boldsymbol{y} = \boldsymbol{A}^T\boldsymbol{x}, \qquad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{F}^{(a)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{F}^{(b)} \end{bmatrix}. \tag{11}$$

Now let us partition the matrix $\boldsymbol{A}$ as follows, where $\boldsymbol{A}_c$ contains the transition filters ($\boldsymbol{F}_5^{(a)}$ and $\boldsymbol{F}_1^{(b)}$):

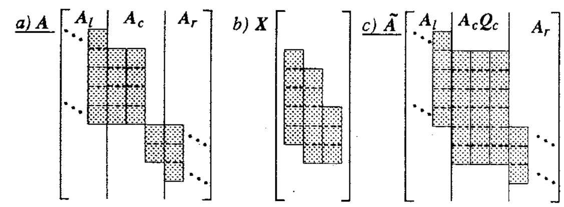
$$\boldsymbol{A} = [\boldsymbol{A}_l|\boldsymbol{A}_c|\boldsymbol{A}_r]. \tag{12}$$

For the optimized transform matrix, we use the structure

$$\tilde{\boldsymbol{A}} = [\boldsymbol{A}_l|\boldsymbol{A}_c\boldsymbol{Q}_c|\boldsymbol{A}_r]. \tag{13}$$

This means that we rotate the last columns of $\boldsymbol{F}^{(a)}$ together with the first columns of $\boldsymbol{F}^{(b)}$ and get overlapping transition filters. Like in section 4.2, we define an objective function of the form

$$\phi_c = \left\|\boldsymbol{L}^T\boldsymbol{X} - \boldsymbol{L}^T\boldsymbol{A}_c\boldsymbol{Q}_c\right\|_F^2 \stackrel{!}{=} \min. \tag{14}$$



Ffigure 4: Transition filters for a switch of filter coefficients (from $L_p = 5$ to $L_p = 3$)

The matrix $\boldsymbol{X}$ contains filter coefficients that lead to our desired subband statistics during transition. The optimal matrix $\boldsymbol{Q}_c$ is given by $\boldsymbol{Q}_c = \boldsymbol{U}\boldsymbol{V}^T$, where $\boldsymbol{U}\Sigma\boldsymbol{V}^T$ is the singular value decomposition of the matrix $\boldsymbol{A}_c^T\boldsymbol{R}_{xx}\boldsymbol{X}$.

5. RESULTS

To give a simple example, a switch from a four-channel to a two-channel paraunitary non-linear phase filter bank will be considered. The four-channel filter bank is obtained by using a tree structure where the high- and low-pass branches of the two-channel filter bank are split into high- and low-pass branches again (using the same filters).

First, a low-pass prototype ($h_0^C(n)$) that leads to minimum cross-correlation between the low- and high-pass branches in the two-channel filter bank will be considered. The coefficients are listed in [5], Table I.

The initial boundary filters for both filter banks were found by Gram-Schmidt orthonormalization. For a joint optimization of all transition filters, $\boldsymbol{X}$ is chosen to be the transition matrix for a direct switch from the four-channel filter bank (a) to the two-channel filter bank (b) so that (using $\boldsymbol{X}$) we could directly switch our bit allocations. The input process is a stationary AR(1)-process with $\rho = 0.95$.[2]

Results are listed in Table I. We see that the optimized boundary filters have much better statistical properties than the initial filters. The values of ϕ_c show that it is worth optimizing the transition filters jointly. This can also be seen in Figure 5, where the variances of the subband signals are shown for an input signal of length $N = 128$ and a switch of the filter bank structure after 64 samples.

The filter bank used in the first example already distributes the energy very symmetrically to the boundaries (for a non-linear phase filter bank). The decrease of ϕ_c due to a joint optimization can be much larger when we switch between non-linear phase filter banks that distribute the energy more unsymmetrically to the boundaries. Figure 6 illustrates this for the 8-tap Daubechies filters.

6. CONCLUSION

In this paper, design methods for support preservative and time-varying filter banks have been presented. Direct

[2] Although we would not switch in practice when the input process is stationary, we can use $\boldsymbol{X}$ for the design of good transition filters.

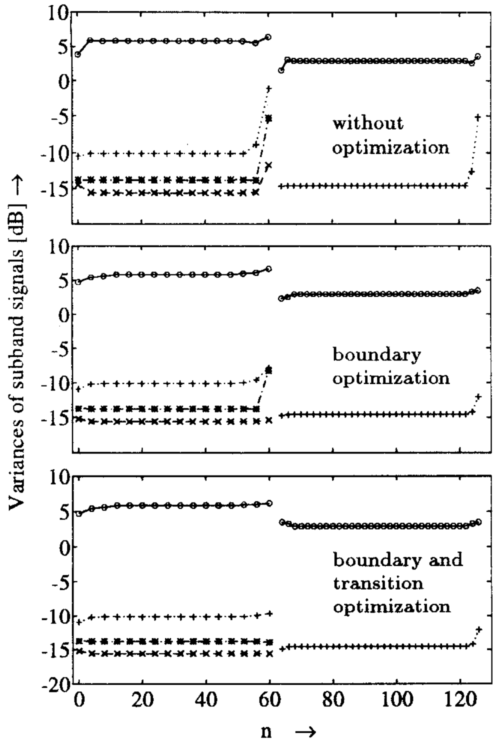


Figure 5: Variances of subband signals in dB. Results for initial and optimized boundary and transition filters ($h_0^C(n)$, AR(1)-process with $\rho = 0.95$)

solutions via SVD to the problem of finding optimal boundary and transition filters have been given. Using these methods, desired statistics at the boundaries and in transition regions can be matched easily. The algorithms can be applied to all FIR filter banks including biorthogonal and pseudo-QMF banks.

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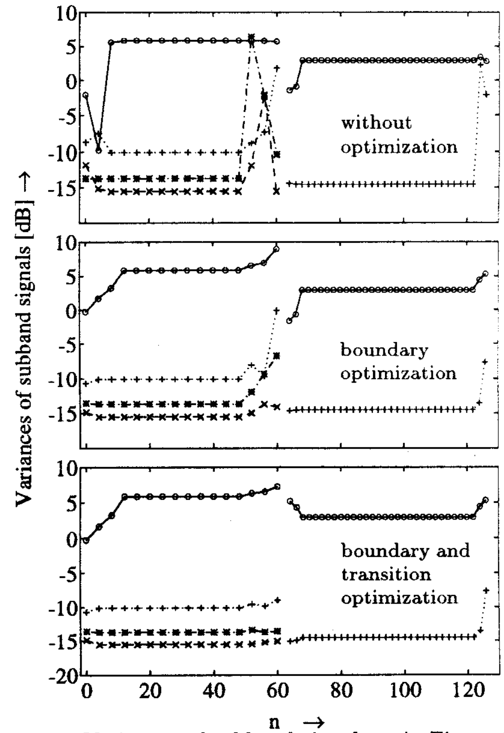

Figure 6: Variances of subband signals as in Figure 5 but for 8-tap Daubechies filters.

TABLE I

RESULTS FOR TWO- AND FOUR-CHANNEL FILTER BANKS BASED ON $h_0^C(n)$ (AR(1)-PROCESS WITH $\rho = 0.95$; E_i/sample ARE AVERAGE ENERGIES OF THE SUBBAND SAMPLES; IN STEADY STATE, WE HAVE E_i/sample= 1); OPTIMIZED VALUES ARE IN BOLDFACE).

	no Opt.	Boundary Opt.	Boundary and Trans. Opt
$E_1^{(a)}$/sample	0.885	0.885	0.885
$E_5^{(a)}$/sample	1.115	1.115	1.042
$E_1^{(b)}$/sample	0.886	0.886	1.108
$E_5^{(b)}$/sample	1.114	1.114	1.082
$\phi_1^{(a)}$	0.55	**0.45**	**0.45**
$\phi_5^{(a)}$	1.05	**0.63**	-
$\phi_1^{(b)}$	0.64	**0.27**	-
$\phi_5^{(b)}$	0.25	**0.19**	**0.19**
ϕ_c	4.65	1.42	**0.83**

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