On the Formulation of the Magnetic Particle Imaging System Function in Fourier Space

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I. Introduction

Several publications on magnetic particle imaging (MPI) are based on the Langevin theory of paramagnetism to describe the imaging process [1-4]. However, although some reconstruction methods require the Fourier transform of the Langevin function and its derivative, in practice, due to the lack of a closed-form expression, they are approximated either numerically or via the Lorentzian function, which works quite well in practice. Nevertheless, we here give a closed-form solution for the Fourier transform of the Langevin function and derive the temporal and spatial Fourier transformed versions of the system function according to the Langevin model.

II. Fourier transform of the Langevin function

The Langevin function has the following uniformly convergent series expansion:

$$\begin{aligned}\mathcal{L}(x) &= \coth(x) - \frac{1}{x} = \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2 + x^2} \\ &= \frac{1}{\mathrm{i}} \sum_{k=1}^{\infty} \left(\frac{1}{k\pi - \mathrm{i}x} - \frac{1}{k\pi + \mathrm{i}x}\right). \end{aligned} \tag{1}$$

First, we derive the Fourier transform of $\mathcal{L}'(x) \in L^1(\mathbb{R})$. We obtain

$$\begin{aligned}\hat{\mathcal{L}}_d(\omega_x) &= \mathcal{F}[\mathcal{L}'(x)] = \sum_{k=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{\mathrm{i}}\left(\frac{1}{k\pi - \mathrm{i}x}\right)'\right] - \mathcal{F}\left[\frac{1}{\mathrm{i}}\left(\frac{1}{k\pi + \mathrm{i}x}\right)'\right]\right) \\ &= \begin{cases} \frac{2\pi|\omega_x|}{e^{\pi|\omega_x|}-1}, & \text{if } |\omega_x| > 0 \\ 2, & \text{if } \omega_x = 0, \end{cases}\end{aligned} \tag{2}$$

where $\omega_x = 2\pi f_x$ denotes the spatial frequency along the x-dimension. With an approach similar to (2), we can calculate the Fourier transform of $\frac{\mathcal{L}(x)}{x}$, which is not in $L^1(\mathbb{R})$, but in $L^2(\mathbb{R})$. This yields

$$\hat{\mathcal{L}}_n(\omega_x) = \mathcal{F}\left[\frac{\mathcal{L}(x)}{x}\right] = -2\ln\left(1 - e^{-\pi|\omega_x|}\right) \in L^1(\mathbb{R}). \tag{3}$$

It should be explicitly noted that the Fourier transform of $\mathcal{L}(x)$ only exists in the sense of the distribution theory, because $\mathcal{L}(x)$ is neither in $L^1(\mathbb{R})$ nor in $L^2(\mathbb{R})$. The distributional Fourier transform of $\mathcal{L}(x)$ is given by

$$\hat{\mathcal{L}}(\omega_x) = \mathcal{F}[\mathcal{L}(x)] = 2\pi\mathrm{i}\,\mathrm{sgn}(\omega_x)\frac{1}{1 - e^{\pi|\omega_x|}} \tag{4}$$

for $\omega_x \neq 0$. The Fourier transform of the Langevin function has a singularity at $\omega_x = 0$. However, it can be easily verified that $\mathcal{L}(x)$ can be expressed as

$$\mathcal{L}(x) = \frac{1}{2}\left(\mathrm{sgn}(x) * \mathcal{L}'(x)\right), \tag{5}$$

where $*$ denotes the convolution operation. The function $\mathrm{sgn}(x)$ can be seen as a temperate distribution from $\mathcal{S}'$ in the spatial domain which also has a distributional expression in the Fourier domain given by $\frac{2}{\mathrm{i}\omega_x}$ [5]. Both distributions are to be understood in the sense that the Fourier transform and its inverse have to be evaluated with help of the Cauchy principal value (p.v.). It can be proofed that this makes the last term in (4) itself a distribution, which can be evaluated with the p.v..

III. 1D MPI Fourier representation of the Langevin-Model

One-dimensional MPI can be described by the simplified model [1]

$$\begin{aligned}u(t) &= \frac{d}{dt}\left[\int_{-\infty}^{\infty} c(x) M_0 \mathcal{L}\left(\beta G(x_{FFP}(t) - x)\right) dx\right] \\ &= \frac{d}{dt}\Phi(t) = \left[\int_{-\infty}^{\infty} c(x)\, s(x,t)\, dx\right]\end{aligned} \tag{6}$$

with $\beta = \frac{\mu_0 m}{k_B T}$ and $M_0 = \mu_0 p m$, where $u(t)$ denotes the measured voltage signal, $c(x)$ is the spatial SPIOs distribution, p denotes the coil sensitivity, μ_0 the vacuum permeability, k_B the Boltzmann's constant, T the temperature of the SPIOs, and m the magnetic moment of one nanoparticle. All parameters in (6) that are independent of $c(x)$ are included in $s(x,t)$, the so-called system function. The spatial position of the FFP at time point t is given by $x_{\mathrm{FFP}}(t) = -G^{-1}H^D(t)$, where $H^D(t)$ denotes the magnetic drive field and G denotes the applied gradient strength of the static gradient field. Commonly, one chooses $H^D(t) = -A\cos(2\pi f t)$, which is a periodic function in $T_D = \frac{1}{f}$. For the sake of clarity, we omit the constant factor M_0 in the following. The Fourier series coefficients of $u(t)$ are given by

$$\hat{u}_k = \frac{1}{T_D}\int_{-T_D/2}^{T_D/2} \frac{d}{dt}\Phi(t) e^{-\mathrm{i}\omega_k t} dt = \frac{\mathrm{i}\,\omega_k}{T_D}\int_{-T_D/2}^{T_D/2} \Phi(t) e^{-\mathrm{i}\omega_k t} dt, \tag{7}$$

where $\omega_k = 2\pi k f$. A convolution in the spatial domain is a multiplication in the frequency domain:

$$h(x) = \int_{-\infty}^{\infty} c(u)\mathcal{L}(\beta G(x-u))du$$
$$\hat{h}(\omega_x) = \hat{c}(\omega_x)\frac{1}{|\beta G|}\hat{\mathcal{L}}\left(\frac{\omega_x}{\beta G}\right). \qquad (8)$$

We are now interested in the Fourier series expansion of $g(t) = h(x_{\mathrm{FFP}}(t))$. In a first step, $h(x_{\mathrm{FFP}}(t))$ is represented by the inverse Fourier transform [6] along *x*:

$$h(x_{\mathrm{FFP}}(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{h}(\omega_x)\, e^{\mathrm{i}\omega_x x_{\mathrm{FFP}}(t)} d\omega_x. \qquad (9)$$

Then, we derive the Fourier series coefficients of $g(t)$:

$$\hat{g}_k = \frac{1}{T_D}\int_{-\frac{T_D}{2}}^{\frac{T_D}{2}} h(x_{\mathrm{FFP}}(t))e^{-\mathrm{i}\omega_k t}dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(x_{\mathrm{FFP}}\left(\frac{z}{2\pi f}\right)\right)e^{-\mathrm{i}kz}\,dz \qquad (10)$$
$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty}\hat{h}(\omega_x)\, e^{\mathrm{i}\omega_x x_{\mathrm{FFP}}\left(\frac{z}{2\pi f}\right)}d\omega_x\right)e^{-\mathrm{i}kz}dz.$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{h}(\omega_x)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}\left(-\omega_x x_{\mathrm{FFP}}\left(\frac{z}{2\pi f}\right)+kz\right)}\,dz\right)d\omega_x.$$

We now consider the function

$$P(\omega_x, k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}\left(-\omega_x x_{\mathrm{FFP}}\left(\frac{z}{2\pi f}\right)+kz\right)}\,dz$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}\left(-\frac{\omega_x A}{G}\cos(z)+kz\right)}\,dz. \qquad (11)$$

The integration problem can be solved with help of the Jacobi-Anger expansion:

$$e^{\mathrm{i}\omega\cos(z)} = \sum_{n=-\infty}^{\infty}\mathrm{i}^n J_n(\omega)e^{\mathrm{i}nz}, \qquad (12)$$

where $J_n(\omega)$ denotes the *n*-th Bessel function of first kind. With (12) it follows

$$P(\omega_x, k) = \mathrm{i}^k J_k\left(\frac{\omega_x A}{G}\right). \qquad (13)$$

Finally, we obtain for (10)

$$\hat{g}_k = \frac{\mathrm{i}^k}{2\pi|\beta G|}\int_{-\infty}^{\infty}\hat{c}(\omega_x)\hat{\mathcal{L}}\left(\frac{\omega_x}{\beta G}\right)J_k\left(\frac{\omega_x A}{G}\right)d\omega_x\,. \qquad (14)$$

It should be noted that the product of $\hat{\mathcal{L}}(\omega_x)$ and $J_k(\alpha\omega_x)$ is always in $L^1(\mathbb{R})$ for $k > 0$ and $\alpha \neq 0$. Next, we derive the Fourier representation of the MPI system equation (7) as

$$\hat{u}_k = \frac{\mathrm{i}^{k+1}\,\omega_k}{2\pi|\beta G|}M_0\int_{-\infty}^{\infty}\hat{c}(\omega_x)\hat{\mathcal{L}}\left(\frac{\omega_x}{\beta G}\right)J_k\left(\frac{\omega_x A}{G}\right)d\omega_x$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}_k(-\omega_x)\hat{c}(\omega_x)d\omega_x = \int_{-\infty}^{\infty}s_k(x)c(x)dx, \qquad (15)$$

where $\hat{s}_k(\omega_x)$ with $k \in \mathbb{N}$ and $\omega_x \in \mathbb{R}$ is the 2D frequency-domain representation of the spatio-temporal system function. Another term needed is the inverse Fourier transform of $J_k(\omega_x)$ along ω_x. The result can be derived as [7]

$$j_k(x) = \mathcal{F}^{-1}[J_k(\omega_x)] = \begin{cases} \frac{\mathrm{i}^k T_k(x)}{\pi\sqrt{1-x^2}} & \text{for } |x| < 1 \\ 0 & \text{else,} \end{cases} \qquad (16)$$

where $T_k(x)$ denotes the k-th Chebyshev polynomial of the first kind. Finally, we want to show that the Fourier representation

$$\hat{s}_k(\omega_x) = \frac{(-\mathrm{i})^{k+1}\,\omega_k M_0}{|\beta G|}\hat{\mathcal{L}}\left(\frac{\omega_x}{\beta G}\right)J_k\left(\frac{\omega_x A}{G}\right) \qquad (17)$$

is consistent with the closed-form solution for $s_k(x)$. Let us remove the spatial Fourier transform to get

$$s_k(x) = \frac{(-\mathrm{i})^{k+1}\,\omega_k\, M_0 G}{A}\int_{-\frac{A}{G}}^{\frac{A}{G}}\mathcal{L}(\beta G(x-u))j_k\left(\frac{Gu}{A}\right)\,du$$
$$= -\mathrm{i}2fM_0\mathcal{L}(\beta G x) * \frac{G\,k\,T_k\left(\frac{Gx}{A}\right)}{A\sqrt{1-\left(\frac{Gx}{A}\right)^2}}$$
$$= \mathrm{i}2fM_0\mathcal{L}(\beta G x) * \frac{\partial}{\partial x}\left(U_{k-1}\left(\frac{Gx}{A}\right)\sqrt{1-\left(\frac{Gx}{A}\right)^2}\right) \qquad (18)$$
$$= \mathrm{i}2fM_0\beta G\mathcal{L}'(\beta G x) * U_{k-1}\left(\frac{Gx}{A}\right)\sqrt{1-\left(\frac{Gx}{A}\right)^2},$$

where $U_n(x)$ denotes the n-th Chebyshev polynomial of the second kind, which is equivalent to the closed-form solution from [1].

IV. Conclusions

The description developed in this contribution can be extended to two- and three-dimensional MPI in a similar way with some minor assumptions. We hope this will help us to prove some systemically made observations in MPI, like the nonlinear frequency mixing [8], which connects the spatial frequency in two and more dimensions with the temporal frequency. However, the proof of the nonlinear frequency mixing is still pending.

ACKNOWLEDGEMENTS

This work was supported by the German Research Foundation under grant number ME 1170/7-1.

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