PROCESSING ARBITRARY-LENGTH SIGNALS WITH MDFT FILTER BANKS

Tanja Karp[1], *Jörg Kliewer*[2], *Alfred Mertins*[1], *and Norbert J. Fliege*[1]

[1] Hamburg University of Technology, Telecommunications Institute, 21071 Hamburg, Germany
[2] University Kiel, Institute for Network and System Theory, 24143 Kiel, Germany

ABSTRACT

In this paper, methods for processing arbitrary-length input signals with MDFT filter banks will be presented. The MDFT filter bank can be regarded as the most general type of a special class of filter banks. These filter banks have linear phase analysis filters, but different centers of symmetry due to subsampling with and without a phase shift [1], [2]. Already known extension methods cannot be applied to these filter banks in their original form [3]. We first discuss the symmetric extension for two special input signal lengths. These cases are then incorporated in the general solution for arbitrary-length input signals.

1. MDFT FILTER BANKS

The MDFT filter bank depicted in Figure 1 is a modified M channel complex modulated filter bank with a complex valued input signal $x(n)$ providing perfect reconstruction if the linear phase prototype $h(n)$ is designed properly [1].

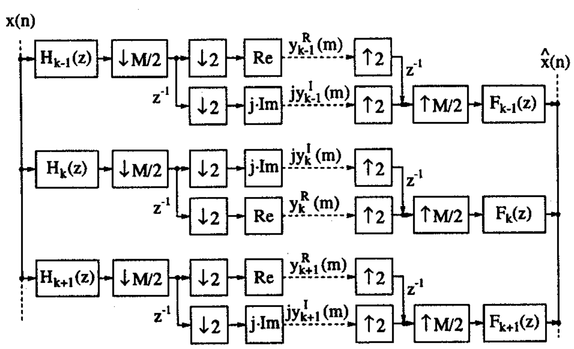

Figure 1: Modified complex modulated filter bank. The real and imaginary subband signals are sampled phase shifted and their phase is alternating between two adjacent subbands.

All linear phase analysis and synthesis filters are derived from the prototype $h(n)$ by complex modulation according to

$$h_k(n) = f_k(n) = h(n) \cdot e^{j\frac{2\pi}{M}k(n-(N-1)/2)},$$
$$\text{with} \quad k = 0, \dots, M-1, \quad n = 0, \dots, N-1,$$

where N denotes the filter length.

The real and imaginary subband signals $y_k^R(m)$ and $j\,y_k^I(m)$, $k = 0, \dots, M-1$, are decimated with and without a phase shift, resp., where in two adjacent subbands the phase of the real and imaginary samples alternate. For *infinite*-length input signals the subband signals are critically subsampled.

2. SYMMETRIC EXTENSION FOR CERTAIN SIGNAL LENGTHS

In this section we regard how critical subsampling can be obtained for *finite*-length input signals by adaptation of the extension methods in [4], [5] to the MDFT filter bank.

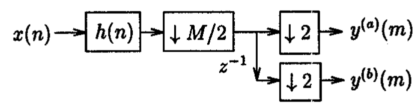

Figure 2: Convolution of the input signal with a real valued linear phase filter and subsampling with two different phase offsets

In a first step, we consider the system depicted in Figure 2, which consists of a real valued linear phase filter $h(n)$ and two $M:1$ decimators operating with a phase offset of $M/2$. This system is equivalent to the lowpass subband of the MDFT analysis filter bank, without consideration of taking the real and imaginary part, respectively. In Sections 2.1 and 2.2 we show for two particular cases, that the subsampled signals $y^{(a)}(m)$ and $y^{(b)}(m)$ show internal symmetries when an extended input signal is applied to the system. In Section 2.3 the methods will be applied to MDFT filter banks.

2.1. Odd Filter Length, Signal Length $N_x = m\,M + 1$

We here discuss the case where the linear phase filter $h(n)$ has odd length (Type 1, [6]) and the complex input signal has length $N_x = mM + 1$, $m \in \mathbb{N}$. This input signal is extended by $Q = \lambda M - (N-1)/2$, $\lambda M > N$, $\lambda \in \mathbb{N}$ samples according to

$$\mathbf{x}_{ex} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{I}_{N_x} \\ \mathbf{E}_2 \end{bmatrix} \mathbf{x} \qquad (1)$$

$$\text{with} \quad \mathbf{x} = [x(0), x(1), \dots, x(N_x - 1)]^T \in \mathbb{C}^{N_x}$$

where $\mathbf{I}_{N_x}$ denotes the $N_x \times N_x$ identity matrix and $\mathbf{E}_1$ and $\mathbf{E}_2$ the size $Q \times N_x$ reflection matrices

$$[\mathbf{E}_1]_{i,j} = \delta_{Q-i,j} \text{ and } [\mathbf{E}_2]_{i,j} = \delta_{i,N_x-2-j},$$
$$i = 0, \dots, Q-1, \quad j = 0, \dots N_x - 1$$

with $\delta_{i,j}$ being the Kronecker symbol.

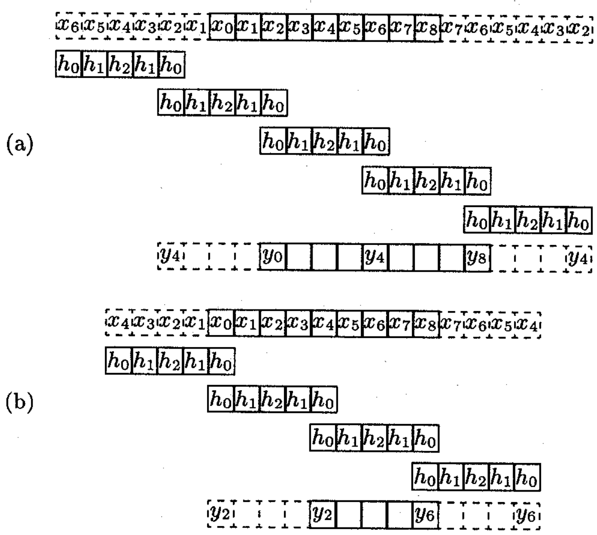

Figure 3: Symmetries of the decimated output signals using an odd length linear phase filter and an input signal with length $N_x = mM + 1$, $m \in \mathbb{N}$.

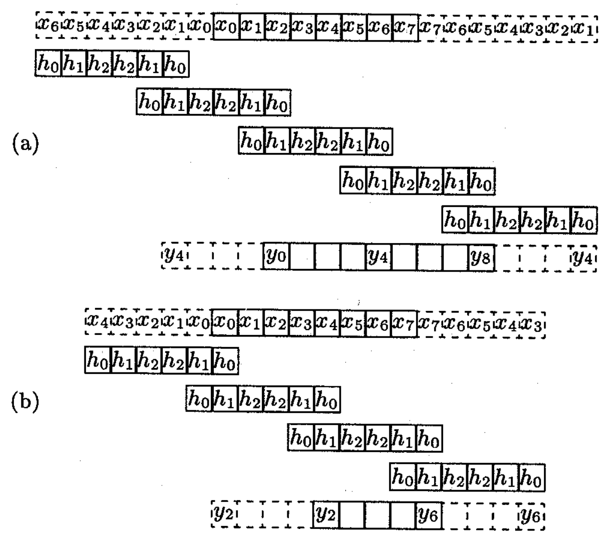

Figure 4: Symmetries of the decimated output signals using an even length linear phase filter and an input signal with length $N_x = mM$, $m \in \mathbb{N}$.

The example in Figure 3 considers a decimation factor of $M = 4$, an input signal length of $N_x = 9$ and a filter length of $N = 5$. In Figure 3a, the result of the convolution in the upper branch of Figure 2 is shown, where the subsampling by factor 4 is already introduced in the shift of the reversed filter impulse response $h(-n)$. The output signal $y^{(a)}(mM)$ has odd symmetry $(\ldots y_4 y_0 y_4 \ldots$ and$\ldots y_4 y_8 y_4 \ldots)$, so that the part in the solid box contains all signal information. Figure 3b shows the delayed branch of Figure 2, where both the subsampling factor 4 and the phase offset of 2 are considered. Here the output signal $y^{(b)}(mM + M/2)$ has even symmetry $(\ldots y_2 y_2 \ldots$ and $\ldots y_6 y_6 \ldots)$, so that it is possible to reduce the signal to the non-redundant part in the solid box, too.

2.2. Even Filter Length, Signal Length $N_x = m\,M$

The input signal is extended as in (1), but Q, $\mathbf{E}_1$, and $\mathbf{E}_2$ have to be chosen as $Q = \lambda M - N/2 + 1$, with $\lambda M > N$, $\lambda \in \mathbb{N}$ and

$$[\mathbf{E}_1]_{i,j} = \delta_{Q-1-i,j}, \quad [\mathbf{E}_2]_{i,j} = \delta_{i,N_x-1-j},$$

$$i = 0, \ldots, Q-1, \quad j = 0, \ldots N_x - 1.$$

Figure 4 shows the resulting symmetries when convolving this input signal with $h(n)$ in Figure 2. Note that in order to achieve symmetry in the output signals the input signal has to be extended in an even style, that means, the first and the last value of the input signal have to be repeated. This was not the case when using odd length filters. However, the resulting symmetries in the upper and lower branch are the same as in the odd length filter case.

2.3. Application to MDFT filter banks

Processing input signals of length $N_x = mM{+}1$ or $N_x = mM$ with the MDFT analysis filter bank using an odd or even length prototype, resp., results in the subband matrices shown in Figure 5. The rows of $\mathbf{Y}_1$ contain all undelayed and those of $\mathbf{Y}_2$ contain all delayed subband samples, where the $\pm$ signs, $y^{R}_{\substack{M/2\\0}}(0)$, and $y^{R}_{\substack{M/2\\0}}(m)$ are valid for $\substack{\text{odd}\\\text{even}}$ length filters.

The zeroth row of $\mathbf{Y}_1$ has the same symmetry as in Figures 3a and 4a and the symmetry in the zeroth row of $\mathbf{Y}_2$ is identical to the symmetries in Figures 3b and 4b.

The $2N_x$ samples in the dashed boxes are the non-redundant part of the real or imaginary subband signals; they contain all information necessary for reconstructing the input signal without boundary distortions, guaranteeing support preservation.

Outside the dashed boxes, the k-th row is extended with the flipped positive or negative samples of the $(M-k)$-th row, $k = 1, \ldots, M-1$. These unusual symmetries are due to the relation

$$h_k(n) = \pm h^*_{M-k}(n), \quad k = 1, \ldots, M-1.$$

3. SYMMETRIC EXTENSION FOR ARBITRARY FILTER AND SIGNAL LENGTHS

3.1. Problem Statement

The symmetries in the subband signals y in Figure 5 only hold for the special cases explained in Section 2. In order to recover these symmetries for arbitrary input signal lengths, $x(n)$ has to be extended with — at this point arbitrary — "don't care" values before the reflection in (1) is carried

$$\mathbf{Y}_1 = \left[\begin{array}{ccccccccc}
\cdots & y_0^R(1) & y_0^R(0) & y_0^R(1) & \cdots & y_0^R(m-1) & y_0^R(m) & y_0^R(m-1) & \cdots \\
 & \pm jy_{M-1}^I(1) & jy_1^I(0) & jy_1^I(1) & & jy_1^I(m-1) & jy_1^I(m) & \pm jy_{M-1}^I(m-1) & \\
\cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \pm y_{M-2}^R(m-1) & \cdots \\
 & & jy_{M/2-1}^I(0) & jy_{M/2-1}^I(1) & & jy_{M/2-1}^I(m-1) & jy_{M/2-1}^I(m) & \vdots & \\
 & \pm y_{M/2}^R(1) & \underset{0}{y_{M/2}^R(0)} & y_{M/2}^R(1) & \cdots & y_{M/2}^R(m-1) & \underset{0}{y_{M/2}^R(m)} & \pm y_{M/2}^R(m-1) & \\
\cdots & \vdots & \pm jy_{M/2-1}^I(0) & jy_{M/2+1}^I(1) & \cdots & jy_{M/2+1}^I(m-1) & \pm jy_{M/2-1}^I(m) & \vdots & \cdots \\
 & & \vdots & \vdots & & \vdots & & & \\
 & \pm jy_1^I(1) & \pm jy_1^I(0) & jy_{M-1}^I(1) & \cdots & jy_{M-1}^I(m-1) & \pm jy_1^I(m) & \pm jy_1^I(m-1) &
\end{array}\right]$$

$$\mathbf{Y}_2 = \left[\begin{array}{cccccccc}
\cdots & jy_0^I(0) & jy_0^I(0) & jy_0^I(1) & \cdots & jy_0^I(m-1) & jy_0^I(m-1) & \cdots \\
 & \pm y_{M-1}^R(0) & y_1^R(0) & y_1^R(1) & & y_1^R(m-1) & \pm y_{M-1}^R(m-1) & \\
 & & \vdots & \vdots & & \vdots & & \\
\cdots & \pm jy_2^I(0) & jy_{M-2}^I(0) & jy_{M-2}^I(1) & & jy_{M-2}^I(m-1) & \pm jy_2^I(m-1) & \cdots \\
 & \pm y_1^R(0) & y_{M-1}^R(0) & y_{M-1}^R(1) & \cdots & y_{M-1}^R(m-1) & \pm y_1^R(m-1) &
\end{array}\right]$$

Figure 5: Matrices containing the undelayed and delayed subband signals for the cases discussed in Section 2. The dashed boxes show the non-redundant samples.

out. Thus we obtain a new complex input vector

$$\tilde{\mathbf{x}} = [x_0, \dots, x_{N_x-1}, a_0, \dots, a_{N_a-1}]^T, \qquad (2)$$

with $N_a = N_{\tilde{x}} - N_x$ and

$$N_{\tilde{x}} = \begin{cases} \tilde{m}M/2+1, & \tilde{m} = \left\lfloor \frac{2(N_x-2)}{M} \right\rfloor + 1 \\ & \text{for odd filter length,} \\ \tilde{m}M/2, & \tilde{m} = \left\lfloor \frac{2(N_x-1)}{M} \right\rfloor + 1 \\ & \text{for even filter length.} \end{cases}$$

If $\tilde{m}$ is even, this leads us again to the subband matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ in Figure 5. If $\tilde{m}$ is odd, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ have the symmetries depicted in Figure 6. On the right-hand side $\mathbf{Y}_1$ shows now an even instead of an odd symmetry, whereas the symmetry of $\mathbf{Y}_2$ has changed on the left-hand side.

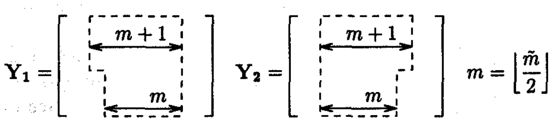

Figure 6: Symmetries of the subband matrices for odd $\tilde{m}$

Critical subsampling is achieved by forcing N_a zero values into the last columns in the dashed boxes of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ by appropriate selection of the inserted values $\mathbf{a} = [a_0, \cdots, a_{N_a-1}]^T$. Thus, the total number of purely real or imaginary valued non-redundant samples in the dashed boxes is reduced to $2N_x$ (i.e. support preservation).

3.2. Solution

In the following we show that the values a_k can be determined by solving a system of linear equations for the elements of the vector

$$\begin{bmatrix} \mathbf{a}^R \\ j\,\mathbf{a}^I \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{\mathbf{a}\} \\ j\,\mathrm{Im}\{\mathbf{a}\} \end{bmatrix},$$

where only the case with odd filter lengths and $\tilde{m}$ even will be regarded. However, the solutions for the remaining three cases are straightforward. As a starting point we define the end of $\mathbf{x}$ as a separate vector $\mathbf{x}_e$ according to

$$\begin{bmatrix} \mathbf{x}_e^R \\ j\mathbf{x}_e^I \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\left\{[x(N_x - \frac{(N-1)+M}{2}), \cdots, x(N_x-1)]^T\right\} \\ j\mathrm{Im}\left\{[x(N_x - \frac{(N-1)+M}{2}), \cdots, x(N_x-1)]^T\right\} \end{bmatrix}$$

The vector $\mathbf{y}_1 = [y_0^R(m), \dots, y_{M/2}^R(m)]^T$ containing the non-redundant values of the m-th column of $\mathbf{Y}_1$ in Figure 5 can now be calculated as

$$[\mathbf{H}_{11}^u\, \mathbf{H}_{12}^u] \begin{bmatrix} \mathbf{x}_e^R \\ \mathbf{a}^R \end{bmatrix} + [\mathbf{H}_{21}^u\, \mathbf{H}_{22}^u] \begin{bmatrix} j\mathbf{x}_e^I \\ j\mathbf{a}^I \end{bmatrix} = \mathbf{y}_1, \qquad (3)$$

$$\mathbf{H}_j^u = \left[\mathbf{H}_{j1}^u\, \mathbf{H}_{j2}^u\right] \in \mathcal{C}^{M/2 \times (N+M+1)/2}, \quad j \in \{1, 2\},$$

where $\mathbf{H}_{j1}^u$ contains the first $(N+M+1)/2 - N_a$ columns of $\mathbf{H}_j^u$ and $\mathbf{H}_{j2}^u$ contains the remaining N_a column vectors. The elements of $\mathbf{H}_j^u$ are defined as

$$[\mathbf{H}_j^u]_{kl} = \begin{cases} \frac{1}{2}\mathrm{Re}\{h_k(l - M/2) + h_k(N-1-l+M/2)\} \\ \qquad \text{for} \begin{cases} j=1, & k \text{ even,} \\ j=2, & k \text{ odd,} \end{cases} \\ \frac{1}{2}j\,\mathrm{Im}\{h_k(l-M/2) + h_k(N-1-l+M/2)\} \\ \qquad \text{for} \begin{cases} j=1, & k \text{ odd,} \\ j=2, & k \text{ even.} \end{cases} \end{cases}$$

For the delayed subbands, we can calculate the vector $\mathbf{y}_2$ containing the $(m-1)$-th column of $\mathbf{Y}_2$ in Figure 5 in the same way.

$$[\mathbf{H}_{11}^d\, \mathbf{H}_{12}^d] \begin{bmatrix} \mathbf{x}_e^R \\ \mathbf{a}^R \end{bmatrix} + [\mathbf{H}_{21}^d\, \mathbf{H}_{22}^d] \begin{bmatrix} j\mathbf{x}_e^I \\ j\mathbf{a}^I \end{bmatrix} = \mathbf{y}_2, \quad (4)$$

where $\mathbf{H}_j^d \in \mathbb{C}^{M\times(N+M+1)/2}$ is composed of $\mathbf{H}_{j1}^d$ and $\mathbf{H}_{j2}^d$ in the same way as $\mathbf{H}_j^u$ and its elements are defined as

$$[\mathbf{H}_j^d]_{kl} = \begin{cases} \frac{1}{2}\mathrm{Re}\{h_k(l-M) + h_k(N-1-l+M)\} \\ \qquad \text{for} \begin{cases} j=1, & k \text{ even}, \\ j=2, & k \text{ odd}, \end{cases} \\ \frac{1}{2}j\,\mathrm{Im}\{h_k(l-M) + h_k(N-1-l+M)\} \\ \qquad \text{for} \begin{cases} j=1, & k \text{ odd}, \\ j=2, & k \text{ even}. \end{cases} \end{cases}$$

In order to achieve support preservation we have to force N_a values out of each vector $\mathbf{y}_1$ and $\mathbf{y}_2$ to zero. We have the degree of freedom to select N_a rows out of (3) and (4), since the lengths of $\mathbf{y}_1$ and $\mathbf{y}_2$ are larger than N_a. This can be carried out by multiplying the equations (3) and (4) with appropriate matrices $\mathbf{S}_1 \in \mathbb{R}^{N_a\times M/2}$ and $\mathbf{S}_2 \in \mathbb{R}^{N_a\times M}$, respectively. $\mathbf{S}_1$ and $\mathbf{S}_2$ contain zeros except a single '1' in each row, which is placed in the ℓ-th column if we want to select the ℓ-th row of (3) or (4), resp. Taking this into account and combining (3) and (4) yields the desired system of linear equations for the inserted values a_k:

$$\begin{bmatrix} \mathbf{S}_1\mathbf{H}_{12}^u & \mathbf{S}_1\mathbf{H}_{22}^u \\ \mathbf{S}_2\mathbf{H}_{12}^d & \mathbf{S}_2\mathbf{H}_{22}^d \end{bmatrix} \begin{bmatrix} \mathbf{a}^R \\ j\,\mathbf{a}^I \end{bmatrix} = -\begin{bmatrix} \mathbf{S}_1\mathbf{H}_{11}^u & \mathbf{S}_1\mathbf{H}_{21}^u \\ \mathbf{S}_2\mathbf{H}_{11}^d & \mathbf{S}_2\mathbf{H}_{21}^d \end{bmatrix} \begin{bmatrix} \mathbf{x}_e^R \\ j\,\mathbf{x}_e^I \end{bmatrix} \quad (5)$$

The values a_k are not necessarily in the same range as the input signal $x(n)$. This might cause problems in real time implementations on DSPs with fixed-point arithmetic.

Note, that in the case of a real input signal (5) reduces to

$$\begin{bmatrix} \mathbf{S}_1 \cdot \mathbf{H}_{12}^u \\ \mathbf{S}_2 \cdot \mathbf{H}_{12}^d \end{bmatrix} \mathbf{a}^R = -\begin{bmatrix} \mathbf{S}_1 \cdot \mathbf{H}_{11}^u \\ \mathbf{S}_2 \cdot \mathbf{H}_{11}^d \end{bmatrix} \mathbf{x}_e^R.$$

Instead of extending the input signal at its end as described above one can also think about an extension at its beginning or at both sides. The solutions for these cases are straightforward.

4. AN IMAGE CODING EXAMPLE

Images are typical examples of finite-length input signals. Figure 7a shows the original 250 × 250 pixel image which will be applied to the extension method proposed above. We use a 16 channel MDFT filter bank with analysis filter length $N = 32$ and process rows and columns of the image separately. First, every row of the original image is extended to length 256. After the subband decomposition of the rows, the columns of length 250 can be processed in the same way. We then quantize all subbands with the same uniform quantizer. After application of the synthesis filter bank we yield the image displayed in Figure 7b. Note that ringing effects due to coarse quantization but hardly any boundary distortions are visible in the reconstructed image.

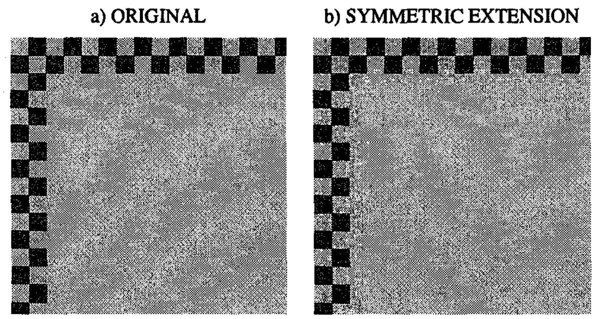

Figure 7: a) Original image and b) Reconstructed image using symmetric extension

5. CONCLUSION

In this paper, we have shown how extension methods can be applied to MDFT filter banks. This enables us to use the MDFT filter bank in applications which deal with finite-length signals like image coding and in general filter banks with time varying properties like number of subbands, transition bandwidth and stopband attenuation of the prototype filters. In the latter case, the input signal (e.g. a wideband speech or audio signal) is divided into finite-length blocks and processed with those filter parameters which yield the best possible reconstruction of the quantized signal.

6. REFERENCES

[1] T. Karp and N. J. Fliege. MDFT Filter Banks with Perfect Reconstruction. In *Proc. ISCAS*, Seattle, USA, May 1995.

[2] Y. Lin and P. P. Vaidyanathan. Linear Phase Cosine Modulated Maximally Decimated Filter Banks with Perfect Reconstruction. In *Proc. ISCAS*, volume 2, pages 17–19, London, UK, June 1994.

[3] T. Karp, A. Mertins, and N. J. Fliege. Processing Finite-Length Signals with MDFT Filter Banks. In *Proc. 15th GRETSI Symposium on Signal and Image Processing*, Juan les Pins, France, September 1995.

[4] M. J. T. Smith and S. L. Eddins. Analysis/Synthesis Techniques for Subband Image Coding. *IEEE Trans. on Acoustics, Speech and Signal Processing*, ASSP-38:1446–1456, August 1990.

[5] L. Chen, T. Nguyen, and K. P. Chan. Symmetric Extension Methods for M-Channel PR LP FIR Analysis/Synthesis Systems. In *Proc. ISCAS*, London, UK, June 1994.

[6] P. P. Vaidyanathan. *Multirate Systems and Filter Banks.* Prentice Hall, Englewood Cliffs, 1993.