LIFTING SCHEMES FOR BIORTHOGONAL MODULATED FILTER BANKS

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ABSTRACT

In this paper we connect the design of biorthogonal modulated filter banks with the idea of lifting schemes, which were proposed for the construction of biorthogonal wavelets. Based on this formulation, we derive a lattice-like structure for the polyphase components. As the rotation-based structures for the design and implementation of paraunitary modulated filter banks, our structure automatically guarantees the PR property of the filter bank. The coefficients can be freely chosen and used for filter optimization.

1. INTRODUCTION

Modulated filter banks are very popular since they provide a very efficient realization. They consist of two main stages: the polyphase filters of the prototype and a discrete cosine or Fourier transform. Figure 1 shows the analysis part of an M-channel cosine modulated filter bank.

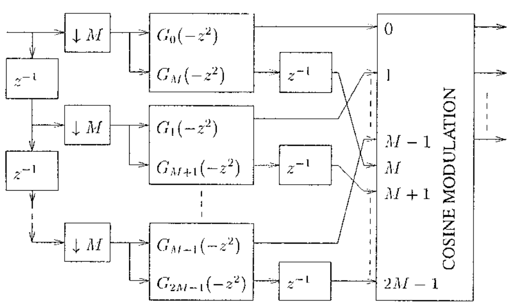

Figure 1: Cosine-modulated M-channel analysis bank

Biorthogonal modulated filter banks (when compared to paraunitary ones) provide the advantage that the overall system delay can be chosen independently of the filter length, thus resulting in low-delay filter banks. They have recently been studied in literature by several authors [1, 2, 3, 4, 5].

While Schuller and Smith describe the filter bank by a cascade of selfinverse sparse matrices and a modulation matrix, the approach derived by Nguyen et al. is based on constraints on the polyphase filters of a common prototype filter for analysis and synthesis.

Based on this formulation for the PR constraints, we here derive a construction scheme for the polyphase filters that results in a lattice-like structure for the polyphase components. Like the rotations proposed in [6] for the case of paraunitary modulated filter banks, our structure automatically guarantees the PR property of the filter bank.

The obtained coefficients can be used for filter optimization, i.e. reduction of the stopband energy of the frequency response, or in order to design filters with integer-valued coefficients.

As shown in [5] the analysis and synthesis filters, $H_k(z)$ and $F_k(z)$, of an M-channel cosine-modulated filter bank can be derived from a prototype $P(z)$ according to

$$h_k(n) = 2p(n)\cos\left[(2k+1)\frac{\pi}{2M}(n-\frac{D}{2}) + (-1)^k\frac{\pi}{4}\right], \quad (1)$$

$$f_k(n) = 2p(n)\cos\left[(2k+1)\frac{\pi}{2M}(n-\frac{D}{2}) - (-1)^k\frac{\pi}{4}\right], \quad (2)$$

The filter length, N, and the overall delay, $D = 2sM + d$ (where $s \in \mathbb{N}$ and $0 \leq d < 2M$), can be chosen arbitrarily. The filter bank is PR if the type-1 polyphase filters $g_k(m) = p(2mM + k)$, $k = 0, \dots, 2M-1$, of the lowpass prototype $P(z)$ satisfy the constraints given in (3)-(6), see [5] for a proof.

Each PR constraint depends on maximally four type-1 polyphase components. The overall system delay can be chosen independently of the filter length. In the following, we restrict ourselves to the case $d = 2M - 1$ and $N = 2mM$ such that the PR constraints reduce to (5) and all $2M$ polyphase filters are of the same length m.

2. THE LIFTING SCHEME

The lifting scheme is a systematic way for the construction of biorthogonal wavelets [7, 8]. One starts with short filters, or simply with the polyphase transform, and successively increases the filter length. A single lifting step consists of increasing the length of one filter from a PR filter pair while keeping the PR property. Such a step is typically followed by a dual lifting step where the length of the second filter is increased. Overall, one alternates lifting and dual lifting to construct long biorthogonal wavelets from short ones.

We here apply the lifting scheme to the polyphase filters $G_k(z)$ of the prototype filter. Thus the filter length can be increased while keeping the filter bank PR.

$$0 \le d < M: \quad G_{d-\ell}(z)G_\ell(z) + z^{-1}G_{d+M-\ell}(z)G_{M+\ell}(z) = \frac{z^{-s}}{2M} \quad \text{for } 0 \le \ell \le d \tag{3}$$

$$z^{-1}G_{d+2M-\ell}(z)G_\ell(z) + z^{-1}G_{d+M-\ell}(z)G_{M+\ell}(z) = \frac{z^{-s}}{2M} \quad \text{for } d+1 \le \ell \le M-1 \tag{4}$$

$$M \le d \le 2M-1: \quad G_{d-\ell}(z)G_\ell(z) + G_{d-M-\ell}(z)G_{M+\ell}(z) = \frac{z^{-s}}{2M} \quad \text{for } 0 \le \ell \le d-M \tag{5}$$

$$G_{d-\ell}(z)G_\ell(z) + z^{-1}G_{d+M-\ell}(z)G_{M+\ell}(z) = \frac{z^{-s}}{2M} \quad \text{for } d+1-M \le \ell \le M-1 \tag{6}$$

2.1. Increasing the Filter Length

Let us start with a given set of four polyphase filters that satisfy the PR constraint according to (5). Using the lifting scheme described in [7, 8] for the construction of biorthogonal wavelets, we can construct longer polyphase filters and therefore longer prototype filters (with – hopefully – better frequency responses) that are still PR and have the same delay as the initial filters.

Writing (5) in matrix form, we obtain

$$[G_\ell(z) \quad G_{M+\ell}(z)] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G_{d-\ell}(z) \\ G_{d-M-\ell}(z) \end{bmatrix} = \frac{z^{-s}}{2M}. \tag{7}$$

New polyphase filters $G_\ell^{new}(z)$ and $G_{d-M-\ell}^{new}(z)$ of increased length can be derived when replacing the identity matrix in (7) by a product of a matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$, where $\mathbf{A}$ and $\mathbf{A}^{-1}$ are of the following type:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ A(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -A(z) & 1 \end{bmatrix} \tag{8}$$

with $A(z)$ being a causal FIR filter. We obtain

$$G_\ell^{new}(z) = G_\ell(z) + A(z)G_{M+\ell}(z), \tag{9}$$

$$G_{d-M-\ell}^{new}(z) = -A(z)G_{d-\ell}(z) + G_{d-M-\ell}(z). \tag{10}$$

Note that the system delay remains unchanged.

In a similar way, new polyphase filters $G_{M+\ell}^{new}(z)$ and $G_{d-\ell}^{new}(z)$ of increased length can be constructed using a so-called dual lifting according to

$$\mathbf{B}\mathbf{B}^{-1} = \begin{bmatrix} 1 & B(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -B(z) \\ 0 & 1 \end{bmatrix} \tag{11}$$

with a causal FIR filter $B(z)$, yielding

$$G_{M+\ell}^{new}(z) = B(z)G_\ell(z) + G_{M+\ell}(z), \tag{12}$$

$$G_{d-\ell}^{new}(z) = G_{d-\ell}(z) - B(z)G_{d-M-\ell}(z). \tag{13}$$

The polyphase filters $G_k(z)$ of a prototype filter of length $N = 2mM$ are all of the same length. Choosing $A(z) = a_0 z^{-1}$ and $B(z) = b_0$ and applying one lifting step and one dual lifting step according to (8) and (11), respectively, the length of all polyphase filters is increased by one.

Iterating this procedure, longer filters can be designed that automatically fulfill the PR constraints of the filter bank (even if we quantize the values a_0 and b_0) and do not increase the overall system delay.

Of course, the presented factorization is not the only one. Another solution is e.g. given when swapping $G_\ell(z)$ and $G_{M+\ell}(z)$ in (7) and $G_{d-\ell}(z)$ and $G_{d-M-\ell}(z)$, respectively. Furthermore, one can increase the polyphase filter length by more than one tap per lifting or dual lifting when using filters $A(z)$ and $B(z)$ of higher order.

Comparing the described lifting with the factorization approach in [3] it turns out that the introduced matrices $\mathbf{A}$ and $\mathbf{B}$ are submatrices of the so-called *Zero-Delay Matrices* in [3].

2.2. Increasing the Filter Bank Delay and the Filter Length

In the previous section, lifting steps were introduced that did not affect the system delay. We now consider the case that both, the filter length and the delay, are increased in a lifting step. For this aim, we define matrices

$$z^{-\alpha}\,\mathbf{C}\,\mathbf{C}^{-1} = \begin{bmatrix} z^{-\alpha} & 0 \\ C(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -C(z) & z^{-\alpha} \end{bmatrix} \tag{14}$$

with $C(z)$ being a causal FIR filter and $z^{-\alpha}$ being the additional delay in (5). Replacing the identity matrix in (7) by $z^{-\alpha}\,\mathbf{C}\,\mathbf{C}^{-1}$ yields the new polyphase filters $G_\ell^{new}(z)$ and $G_{d-M-\ell}^{new}(z)$ of increased length:

$$G_\ell^{new}(z) = z^{-\alpha}G_\ell(z) + C(z)G_{M+\ell}(z), \tag{15}$$

$$G_{d-M-\ell}^{new}(z) = -C(z)G_{d-\ell}(z) + z^{-\alpha}G_{d-M-\ell}(z). \tag{16}$$

For dual lifting, the matrices

$$z^{-\alpha}\,\mathbf{D}\,\mathbf{D}^{-1} = \begin{bmatrix} 1 & D(z) \\ 0 & z^{-\alpha} \end{bmatrix} \begin{bmatrix} z^{-\alpha} & -D(z) \\ 0 & 1 \end{bmatrix} \tag{17}$$

with a causal FIR filter $D(z)$, yield

$$G_{M+\ell}^{new}(z) = D(z)G_\ell(z) + z^{-\alpha}G_{M+\ell}(z), \tag{19}$$

$$G_{d-\ell}^{new}(z) = z^{-\alpha}G_{d-\ell}(z) - D(z)G_{d-M-\ell}(z). \tag{20}$$

Again, we want to increase the length of all polyphase filters by one when applying a lifting and dual lifting step. Choosing $C(z) = c_0$, $D(z) = d_0$, and $\alpha = 1$ yields the desired result and increases the delay in (5) by two ($s^{new} = s + 2$) and the overall system delay by $4M$ samples for each combination of lifting and dual lifting. The matrices $\mathbf{C}$ and $\mathbf{D}$ correspond to submatrices of the so-called *Maximum-Delay Matrices* in [3].

$$[G_\ell(z)\ G_{M+\ell}(z)]\begin{bmatrix}G_{d-\ell}(z)\\G_{d-M-\ell}(z)\end{bmatrix} = \frac{1}{4M}\,[1\ 1]\,\mathbf{F}_0\mathbf{F}_1\prod_j(\mathbf{A}_j\mathbf{B}_j)\prod_i(\mathbf{C}_i\mathbf{D}_i)\prod_i(z^{-2}\mathbf{D}_i^{-1}\mathbf{C}_i^{-1})\prod_j(\mathbf{B}_j^{-1}\mathbf{A}_j^{-1})\hat{\mathbf{F}}_1\mathbf{F}_0\begin{bmatrix}1\\1\end{bmatrix}$$

$$\text{with } \mathbf{F}_0 = \begin{bmatrix}1 & 0\\a_{0,0} & 1\end{bmatrix}\begin{bmatrix}1 & b_{0,0}\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\a_{0,1} & 1\end{bmatrix}, \quad \begin{matrix}\mathbf{F}_1 = \mathbf{CB} & \text{and} & \hat{\mathbf{F}}_1 = z^{-1}\mathbf{F}_1^{-1} & \text{for odd } s,\\ \mathbf{F}_1 = \mathbf{I} & \text{and} & \hat{\mathbf{F}}_1 = \mathbf{I} & \text{for even } s\end{matrix} \tag{18}$$

3. COMPLETENESS OF THE FACTORIZATION

So far, we have shown that longer filters that automatically fulfill the PR constraints can be designed using lifting and dual lifting. The delay may be increased or kept unchanged when applying the lifting scheme. In this section we show that the opposite also holds true: each set of polyphase filters that fulfills the PR constraints (5) can be realized from the lazy wavelet as a consecution of lifting and dual lifting using lifting matrices and filters $A(z)$ to $D(z)$ of the proposed form and slightly different matrices $\mathbf{F}_1$ and $\mathbf{F}_0$ in the last step, see (18).

3.1. Reduction of the Filter Length and the Delay

As long as the delay s is greater than one and the length of the polyphase filters, m, is at least two, we can assume that the given polyphase filters are obtained from shorter ones by lifting and dual lifting with $\mathbf{C}$ and $\mathbf{D}$, respectively.

The original polyphase filters are connected with shorter ones, denoted with superscript sh, according to

$$G_{M+\ell}(z) = d_0 G_\ell(z) + z^{-1}G^{sh}_{M+\ell}(z), \tag{21}$$
$$G_{d-\ell}(z) = z^{-1}G^{sh}_{d-\ell}(z) - d_0 G_{d-M-\ell}(z), \tag{22}$$

and

$$G_\ell(z) = z^{-1}G^{sh}_\ell(z) + c_0 G^{sh}_{M+\ell}(z), \tag{23}$$
$$G_{d-M-\ell}(z) = -c_0 G^{sh}_{d-\ell}(z) + z^{-1}G^{sh}_{d-M-\ell}(z). \tag{24}$$

In order to obtain filters $G^{sh}_{M+\ell}(z)$ and $G^{sh}_{d-\ell}(z)$ of reduced length, we have to ensure that the first tap in the time domain of equations (21)-(24) is equal zero, resulting in

$$d_0 = \frac{g_{M+\ell}(0)}{g_\ell(0)} = -\frac{g_{d-\ell}(0)}{g_{d-M-\ell}(0)} \tag{25}$$

and

$$c_0 = \frac{g_\ell(0)}{g^{sh}_{M+\ell}(0)} = -\frac{g_{d-M-\ell}(0)}{g^{sh}_{d-\ell}(0)}. \tag{26}$$

The condition (25) on the first coefficient of the polyphase filters is the same as the first tap of the PR condition (5) in the time domain:

$$g_{d-\ell}(0)g_\ell(0) + g_{d-M-\ell}(0)g_{M+\ell}(0) = 0 \tag{27}$$

and is thus satisfied. The same holds true for (26) given the PR polyphase filters $G_\ell(z)$, $G_{d-M-\ell}(z)$, $G^{sh}_{M+\ell}(z)$, $G^{sh}_{d-\ell}(z)$.

We now have polyphase filter of length $m-1$ and a shorter delay $s^{sh} = s-2$. This splitting of dual lifting and lifting with $\mathbf{D}$ and $\mathbf{C}$ can be iterated while the polyphase filter length and the delay are at least two.

3.2. Reduction of the Filter Length

As long as the polyphase filter length m is at least two and greater than $(s-1)/2$, we can assume that our polyphase filters are constructed from one tap shorter polyphase filters using lifting and dual lifting with matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. The coefficients b_0 and a_0 can be calculated from

$$G_{d-\ell}(z) = G^{sh}_{d-\ell}(z) - b_0 G_{d-M-\ell}(z), \tag{28}$$
$$G_{M+\ell}(z) = b_0 G_\ell(z) + G^{sh}_{M+\ell}(z), \tag{29}$$

and

$$G_\ell(z) = G^{sh}_\ell(z) + a_0 z^{-1}G^{sh}_{M+\ell}(z), \tag{30}$$
$$G_{d-M-\ell}(z) = -a_0 z^{-1}G^{sh}_{d-\ell}(z) + G^{sh}_{d-M-\ell}(z), \tag{31}$$

In order to obtain $G^{sh}_{d-\ell}(z)$ and $G^{sh}_{M+\ell}(z)$ polyphase filters of length $m-1$ the following relations have to hold true

$$b_0 = -\frac{g_{d-\ell}(m-1)}{g_{d-M-\ell}(m-1)} = \frac{g_{M+\ell}(m-1)}{g_\ell(m-1)} \tag{32}$$

and

$$a_0 = \frac{g_\ell(m-1)}{g^{sh}_{M+\ell}(m-2)} = -\frac{g_{d-M-\ell}(m-1)}{g^{sh}_{d-\ell}(m-2)}. \tag{33}$$

The conditions imposed on the polyphase filter coefficients by (32) and (33) are again nothing else but the PR constraint in the time domain, this time for the last sample.

Again, this procedure can be iterated until we end up with polyphase filters of length two and a delay of $s^{sh} = 1$ if s is odd or with filters of length one if s is even.

3.3. Last Iteration Step

If the original delay s was odd, the iteration steps for the filter length reduction ends with polyphase filters of length two and $s^{sh} = 1$. Using a final lifting and dual lifting step with $\mathbf{F}_1 = \mathbf{C}\cdot\mathbf{B}$ reduces the filter length and the delay again by one. The remaining length one filters can be derived from the lazy transform using matrix $\mathbf{F}_0$ for the last step as described in (18).

4. LATTICE-LIKE STRUCTURE

Connecting lifting and dual lifting to single steps results in the lattice-like structures depicted in Figure 2 (lifting that does not increase the delay) and Figure 3 (lifing that maximally increases the delay).

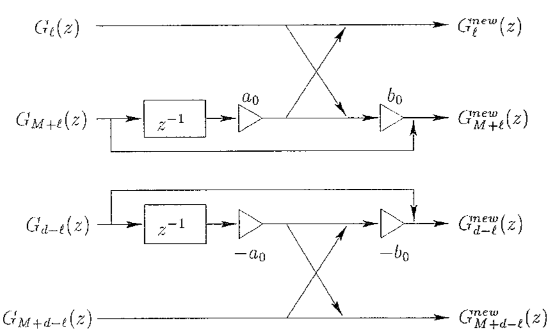

Figure 2: Lattice structure realizing lifting and dual lifting with $A(z) = a_0 z^{-1}$ and $B(z) = b_0$

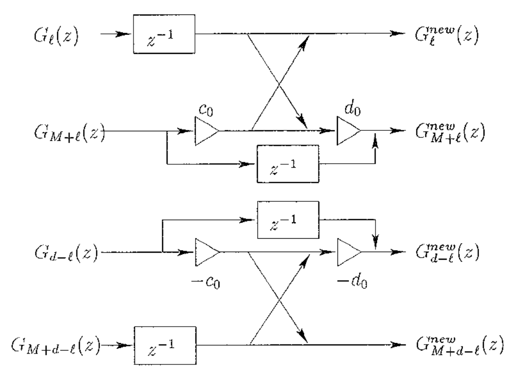

Figure 3: Lattice structure realizing lifting and dual lifting with $C(z) = c_0$ and $D(z) = d_0$ resulting in an increase of the delay by 2 samples

When realizing the polyphase filters in VLSI, the lattice-like structure can be directly used and results in an implementation that is not only extremly regular but also designated for a parallel implementation.

5. DESIGN EXAMPLE

Figure 4 shows the frequency responses for different prototype filters for 8-channel cosine-modulated filter banks. In all cases, the overall system delay is 31 samples. The frequency responses have been obtained using nonlinear optimization starting with a short prototype and successively increasing the length. A significant reduction of the stopband energy can be noticed for longer prototypes.

6. CONCLUSIONS

In this paper we have shown that prototype filters for PR biorthogonal modulated filter banks can be constructed using lifting schemes. When changing the length of the polyphase filters by one in each lifting step a lattice like structure can be derived for the polyphase filters. The

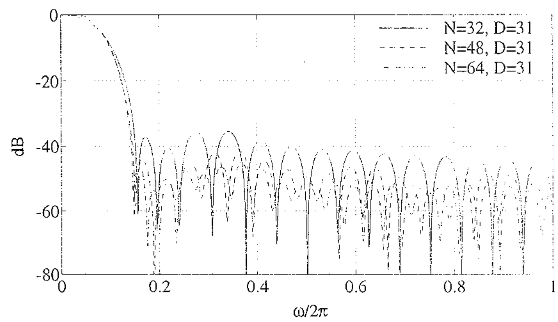

Figure 4: Frequency responses of prototypes for $M = 8$ with different lengths resulting in the same filter bank delay D

coefficients are robust to quantization and the proposed lifting scheme can be used in order to design integer coefficient prototypes.

7. REFERENCES

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