OPTIMAL DYADIC FILTER BANKS FOR SUBBAND CODING AND ZONAL SAMPLING

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Abstract. The design of optimal prototype filters for multiresolution signal decomposition is considered. Filters will be optimized with respect to the following quality criteria: a) maximum coding gain of subband coding over PCM; b) minimum approximation error for zonal sampling. Only in the special case of paraunitary systems both criteria are nearly equivalent. As an example for more general systems, the design of linear phase filters (including linear phase filters with unequal even or odd lengths) will be considered (parametrizations will be briefly shown).

1 Introduction

Recently, multiresolution signal decomposition has been successfully applied to image and audio coding. In statistical optimization of dyadic tree-structured systems, several researchers have focused on paraunitary filter banks [1], [2], [3], [5], [6], [8]. The coding gain of general perfect reconstruction systems has been derived in [4].

Often it is interesting to reconstruct a signal only from a few subbands (zonal sampling). For the two-band paraunitary case it is known that *maximum coding gain* and *minimum approximation error* are equivalent criteria [8] . This paper shows that both criteria are completely different in general (e.g. linear phase filters).

2 PR Filter Bank

Perfect reconstruction (PR) systems are considered throughout this paper. Therefore filters must satisfy the condition ([9])

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2, \tag{1}$$

where $H_0(z)$ and $H_1(z)$ are the analysis filters and $G_0(z)$ and $G_1(z)$ are the synthesis filters.

It can easily be verified that the PR property is guaranteed if the Nyquist-condition

$$H_1(z)G_1(z) + H_1(-z)G_1(-z) = 2 \tag{2}$$

is satisfied and the filters $H_0(z)$ and $G_0(z)$ are chosen as

$$\left.\begin{array}{rcl} H_0(z) &=& z^{2\ell-1}G_1(-z) \\ G_0(z) &=& -z^{-(2\ell-1)}H_1(-z) \end{array}\right\} \quad \ell \in \mathbf{Z}. \tag{3}$$

Moreover, in the paraunitary case we have $H_0(z) = \tilde{G}_0(z)$, where $\tilde{}$ denotes complex conjugation on the unit circle: $\tilde{G}_0(e^{j\Omega}) = (G_0(e^{j\Omega}))^*$.

Usually, the subband signals are computed successively from a given input sequence $c_0(n) = x(n)$ as shown in Figure 1a. However, for the derivation of optimality criteria it is useful to rewrite the subband signals as (see Figure 1b)

$$\begin{array}{rl} c_k(m) &= \sum_\ell x(2^k m - \ell)\ a_k(\ell), \\ d_k(m) &= \sum_\ell x(2^k m - \ell)\ b_k(\ell), \end{array} \tag{4}$$

where the filters are given by

$$\begin{array}{ll} A_1(z) = H_0(z), & A_k(z) = \prod_{i=0}^{k-1} H_0(z^{2^i}), \\ B_1(z) = H_1(z), & B_k(z) = H_1(z^{2^{k-1}}) \prod_{i=0}^{k-2} H_0(z^{2^i}), \\ P_1(z) = G_0(z), & P_k(z) = \prod_{i=0}^{k-1} G_0(z^{2^i}), \\ Q_1(z) = G_1(z), & Q_k(z) = G_1(z^{2^{k-1}}) \prod_{i=0}^{k-2} G_0(z^{2^i}). \end{array} \tag{5}$$

3 Coding Gain

In order to calculate the coding gain of subband coding over PCM for an M-band decomposition, the formula of Katto and Yasada [4] can be used:

$$G_{SBC} = \frac{1}{\prod_{k=1}^{M} (N_k\ \beta_k\ \alpha_k)^{\frac{1}{N_k}}} \tag{6}$$

where

$$\begin{array}{rl} \alpha_k &= \begin{cases} \frac{1}{\sigma_x^2}\mathbf{b}_k^*\mathbf{R}_{xx}^{(k)}\mathbf{b}_k, & k = 1,\dots,M-1 \\ \frac{1}{\sigma_x^2}\mathbf{a}_{M-1}^*\mathbf{R}_{xx}^{(M-1)}\mathbf{a}_{M-1}, & k = M \end{cases} \\ \beta_k &= \begin{cases} \frac{1}{N_k}\left\|\mathbf{q}_k\right\|^2, & k = 1,\dots,M-1 \\ \frac{1}{N_{M-1}}\left\|\mathbf{p}_{M-1}\right\|^2, & k = M \end{cases} \end{array} \tag{7}$$

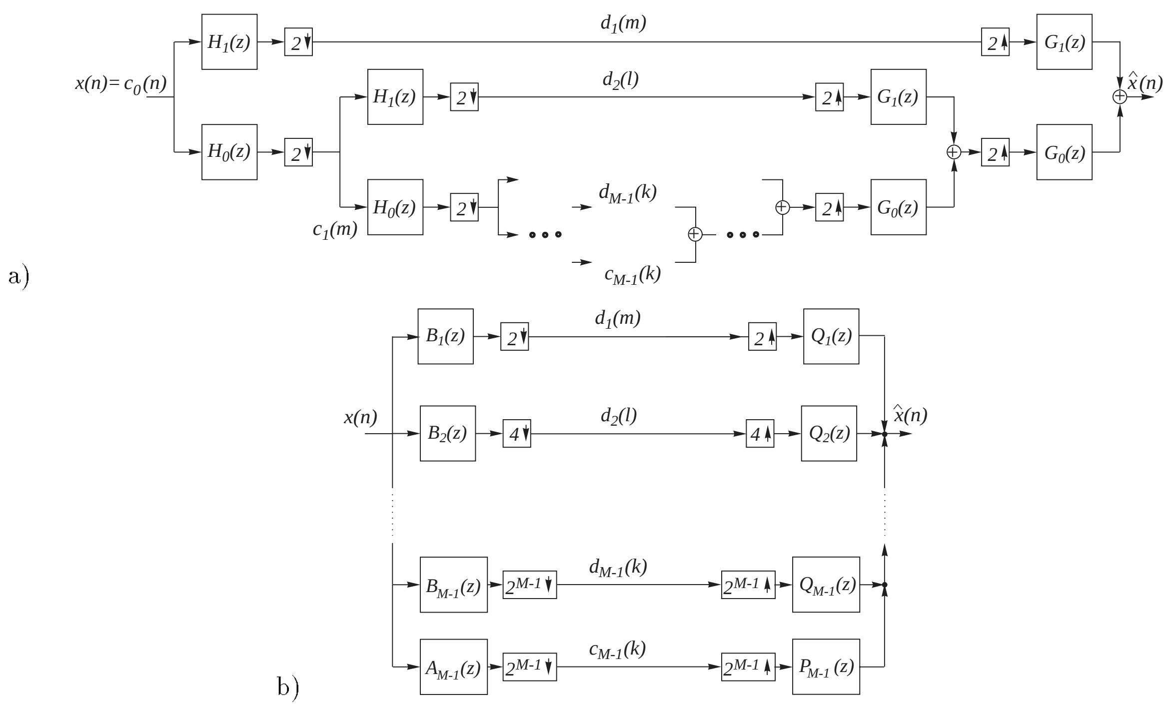

Figure 1: Equivalent filter banks

and

$$N_k = \begin{cases} 2^k, & k = 1, \ldots, M-1 \\ 2^{M-1}, & k = M \end{cases} \tag{8}$$

The asterisk * denotes the hermitian and $\|\cdot\|$ denotes the norm of a vector. The vectors $\mathbf{a}_k, \mathbf{b}_k, \mathbf{p}_k$ and $\mathbf{q}_k$ contain the impulse responses $a_k(n), b_k(n), p_k(n)$ and $q_k(n)$, respectively. $\mathbf{R}_{xx}^{(k)}$ is the $L_k \times L_k$ autocorrelation matrix of the stationary input process $x(n)$, where L_k is the length of the filters $a_k(n)$ and $b_k(n)$, respectively. The entries are $[\mathbf{R}_{xx}^{(k)}]_{i,j} = r_{xx}(j-i) = E\{x(n+j-i)x^*(n)\}$.

4 Zonal Sampling

In most cases where zonal sampling is applied, the high frequency bands will be dropped. If we assume that $M-1$ high frequency bands are not used for synthesis, the quality criterion is

$$\sigma_\eta^2 = E\left\{|x(n) - \hat{x}(n)|^2\right\}, \tag{9}$$

where $\hat{x}(n)$ has to be computed as shown in Figure 2.

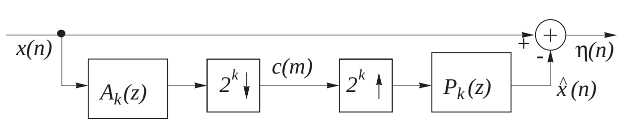

Figure 2: Approximation problem

In the paraunitary case, the signal space is decomposed into an orthogonal sum of subspaces, and the Parseval identity indicates that (9) is equivalent to maximizing the energy $\sigma_c^2 = E\left\{|c(n)|^2\right\}$. In general, the signal space will only be decomposed into a direct sum of subspaces so that the Parseval identity cannot be applied. In order to express the approximation error in terms of $A_k(z)$ and $P_k(z)$, the polyphase decomposition will be used. This method allows the decomposition of the cyclostationary process $\hat{x}(n)$ into 2^k stationary processes at a sampling rate reduced by the factor 2^k.

First, the input and output processes will be decomposed into 2^k polyphase components forming the stationary vector processes

$$\begin{aligned} \mathbf{x}_p(m) &= \left[x(2^k m), x(2^k m+1), \ldots, x(2^k m+2^k-1)\right]^T, \\ \hat{\mathbf{x}}_p(m) &= \left[\hat{x}(2^k m), \hat{x}(2^k m+1), \ldots, \hat{x}(2^k m+2^k-1)\right]^T. \end{aligned} \tag{10}$$

Now the approximation error can be written as

$$\sigma_\eta^2 = 2^{-k} trace\left\{\boldsymbol{\mathcal{R}}_{xx}(0) - \boldsymbol{\mathcal{R}}_{x\hat{x}}(0) - \boldsymbol{\mathcal{R}}_{\hat{x}x}(0) + \boldsymbol{\mathcal{R}}_{\hat{x}\hat{x}}(0)\right\}, \tag{11}$$

where the correlation matrices are defined as

$$\begin{aligned} \boldsymbol{\mathcal{R}}_{xx}(\ell) &= E\left\{\mathbf{x}_p(m+\ell)\mathbf{x}_p^*(m)\right\}, \\ \boldsymbol{\mathcal{R}}_{x\hat{x}}(\ell) &= E\left\{\hat{\mathbf{x}}_p(m+\ell)\mathbf{x}_p^*(m)\right\} = \boldsymbol{\mathcal{R}}_{\hat{x}x}^*(-\ell), \\ \boldsymbol{\mathcal{R}}_{\hat{x}\hat{x}}(\ell) &= E\left\{\hat{\mathbf{x}}_p(m+\ell)\hat{\mathbf{x}}_p^*(m)\right\}. \end{aligned} \tag{12}$$

The correlation matrices can be computed by applying the input-output relation

$$\hat{\mathbf{x}}_p(m) = \boldsymbol{\beta}_k(m) * \boldsymbol{\alpha}_k^*(m) * \mathbf{x}_p(m), \qquad (13)$$

where the vectors $\boldsymbol{\alpha}_k(m)$ and $\boldsymbol{\beta}_k(m)$ contain polyphase components of the impulse responses $a_k(n)$ and $p_k(n)$, respectively. The vectors can be viewed as vector impulse responses that have to be convolved. For causal synthesis and anti-causal analysis filters the definitions are

$$\begin{aligned} \boldsymbol{\alpha}_k^*(n) &= \left[a_k(2^k n), a_k(2^k n-1), \ldots, a_k(2^k n-2^k+1)\right], \\ \boldsymbol{\beta}_k^T(n) &= \left[p_k(2^k n), p_k(2^k n+1), \ldots, p_k(2^k n+2^k-1)\right]. \end{aligned} \qquad (14)$$

From (12) and (13), we then have

$$\begin{aligned} \boldsymbol{\mathcal{R}}_{x\hat{x}}(m) &= \boldsymbol{\beta}_k(m) * \boldsymbol{\alpha}_k^*(m) * \boldsymbol{\mathcal{R}}_{xx}(m), \\ \boldsymbol{\mathcal{R}}_{\hat{x}\hat{x}}(m) &= \boldsymbol{\beta}_k(m) * \boldsymbol{\alpha}_k^*(m) * \boldsymbol{\mathcal{R}}_{xx}(m) * \boldsymbol{\alpha}_k(m) * \boldsymbol{\beta}_k^*(m). \end{aligned} \qquad (15)$$

5 Parametrizations

Parametrizations allowing unconstrained optimization and leading to low-pass filters with a zero at $z = -1$ are used. This condition is necessary if zero mean high-pass filters and a wavelet interpretation are desired. The parametrizations are:

(i) Zous and Tewfiks parametrization for the paraunitary case [7], [3];

(ii) The linear phase lattice [1], which allows the design of linear phase filters having equal even lengths;

(iii) Filter design by solving a linear set of equations. This method allows the design of general perfect reconstruction filter banks including linear phase filters with unequal even or odd lengths.

Given an impulse response $h_1(n)$, the corresponding impulse response $g_1(n)$ may be computed from the linear set of equations

$$\delta_{\ell 0} = \sum_n g_1(n)\; h_1(2\ell - n), \qquad (16)$$

which is the Nyquist-condition (2) in time-domain. Moreover, if zero mean high-pass filters are desired $h_1(n)$ has to be chosen to satisfy $\sum_n h_1(n) = 0$, and the condition $\sum_n g_1(n) = 0$ has to be included in the system of equations (16).

6 Results

The coding gain (6) has been used as an objective function for filter optimization. Some results for AR(1) sources with $\rho = 0.95$, $r_{xx}(\ell) = \rho^{|\ell|}$ are shown in Figure 3. Interestingly, linear phase filters with odd and unequal lengths lead to the best results (3×5 means: 3 coefficients for the analysis high-pass and 5 coefficients for the analysis low-pass filter). A selection of filter coefficients is shown in Table 1.

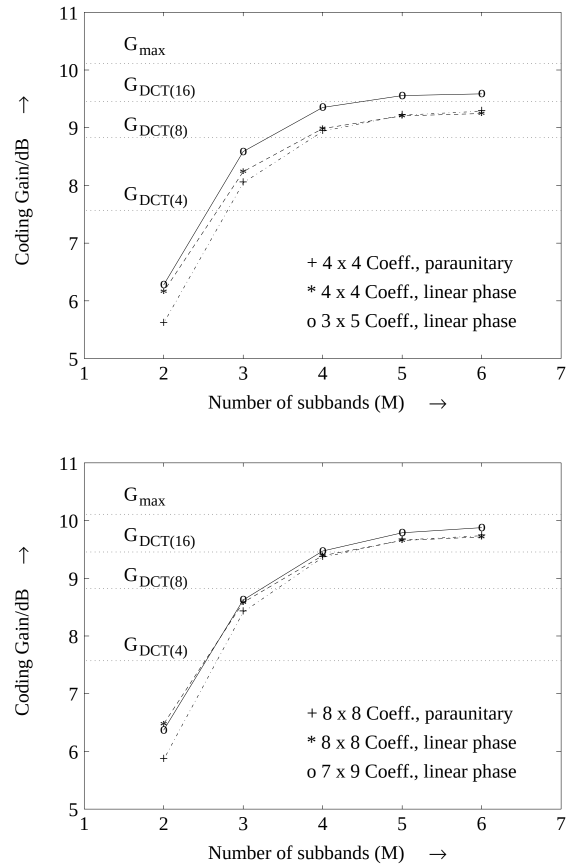

Figure 3: Coding gain of subband coding and DCT (AR(1), $\rho = 0.95$)

Alternatively, the approximation error (11) has been used as an objective function for filter optimization. Some results for AR(1) sources with $\rho = 0.95$ and $\sigma_x^2 = 1$ are shown in Table 2 (L is the filter length). Here the paraunitary filters lead to superior results.

The difference between both criteria can be seen by comparing the frequency responses of the filters shown in Figure 4.

7 Conclusion

The design of optimal prototype filters for multiresolution signal decomposition has been considered. For given input statistics, the filters have been optimized for maximum coding gain and minimum approximation error, respectively. The design methods presented in this paper are not restricted to paraunitary systems. The design of linear phase filters with unequal even or odd

Table 1: Coefficients of Filters with Maximum Coding Gain ($M = 5$)

paraunitary		linear phase			
6 Coeff.	8 Coeff.	5 × 7 Coeff.		7 × 9 Coeff.	
$h_0(n)$	$h_0(n)$	$h_1(n)$	$h_0(n)$	$h_1(n)$	$h_0(n)$
0.38514928	0.06112969	-0.01878374	-0.00620359	-0.07046992	0.04036428
0.79642360	-0.01716949	-0.39548565	-0.13061463	0.04079525	-0.02336700
0.42843242	-0.08925140	0.82853879	0.32227068	0.42507423	-0.11031640
-0.14080794	0.33116666	-0.39548565	0.89336344	-0.79079912	0.37587259
-0.10647492	0.77027527	-0.01878374	0.32227068	0.42507423	0.84491540
0.05149111	0.51788900		-0.13061463	0.04079525	0.37587259
	-0.03504678		-0.00620359	-0.07046992	-0.11031640
	-0.12477939				-0.02336700
					0.04036428
Gain=9.550 dB	Gain=9.663 dB	Gain=9.621 dB		Gain=9.788 dB	

Table 2: Approximation error σ_η^2

	paraunitary		linear phase		linear phase	
$M \downarrow$ $L \rightarrow$	6	8	6	8	3×5	5×7
2	0.0176	0.0171	0.0182	0.0173	0.0184	0.0184
3	0.0422	0.0410	0.0432	0.0416	0.0437	0.0437
4	0.0871	0.0849	0.0886	0.0882	0.0898	0.0896
5	0.1715	0.1678	0.1745	0.1699	0.1769	0.1762

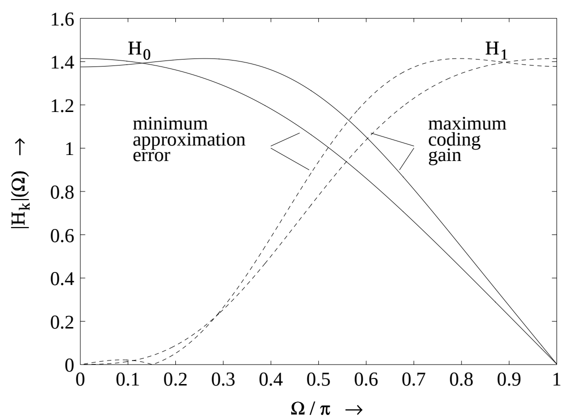

Figure 4: Frequency responses of 6-tap linear phase filters (AR(1), $\rho = 0.95$)

lengths is included. These filters are very interesting because they give the highest coding gain. Paraunitary systems lead to lower approximation errors than linear phase filters do if the high-frequency bands are dropped.

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