LINEAR-PHASE COSINE-MODULATED FILTER BANKS WITHOUT DC LEAKAGE

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ABSTRACT

In this paper, we present a new class of linear-phase cosine-modulated filter banks. In contrast to the $2M$ band structure of the linear-phase DCT-II filter bank derived by Lin and Vaidyanathan, our filter bank has M bands and requires different prototype filters for the analysis and synthesis. Given the PR constraints of the bank, we derive a non-orthogonal lattice structure for implementing the polyphase components of the prototype filters. This structure shares similarities with the one of DCT-IV filter banks and automatically guarantees perfect reconstruction of the bank. It furthermore allows to specify the values of the filters' frequency responses at certain frequencies, thus allowing the design of linear-phase cosine-modulated filter banks without DC leakage. Since analysis and synthesis prototype filters are different, we investigate several cost functions for their design.

1. INTRODUCTION

In image subband coding one prefers linear-phase filters since they allow an easy treatment of the image boundaries [1] and distribute quantization errors symmetrically over the edges in the image.

Due to the very high data rates occurring in image and especially in video coding applications, the computational cost for the signal decomposition and reconstruction must be strictly limited. Modulated filter banks are known to provide a low implementation cost and are also well suited for a parallel implementation since the polyphase filtering of the bank can be done in parallel. The most popular modulation scheme is given by cosine modulation. However, for filter banks based on a DCT-IV modulation it has been shown in [2] that although given a linear-phase prototype filter the analysis and synthesis filters cannot be linear phase.

A cosine-modulated filter bank with linear-phase analysis and synthesis filters which is based on a $2M$ bank structure and uses DCT-II and DST-II modulation has been derived by Lin and Vaidyanathan in [3] and further studied in [4, 5, 6]. One peculiar property of this filter bank is the fact that the center of symmetry for analysis filters being modulated by DCT and those modulated by DST are shifted by M samples. This fact leads to some complications when applying symmetric extension methods in order to provide a support preservative decomposition [7]. For example, the subbands will have a different number of non-redundant samples.

In this paper we present a new class of DCT-II filter banks with linear-phase analysis and synthesis filters being based on a DCT-II modulation and derive constraints for perfect reconstruction of this bank. For reasons of conciseness we restrict ourselves to the case where the filter length N is connected to the number of subbands M by $N = 2rM + M$ with r being an integer and M even. The PR constraints will be given for the general biorthogonal case and then restricted to linear-phase prototypes and filter banks without DC leakage since experiences in image coding have shown that linear-phase filters and perfect reconstruction of the bank are not sufficient in order to obtain a good quality of the reconstructed images at a given compression rate. Artifacts such as the checkerboard effect do not occur if all analysis filters, apart from the lowpass, have a zero at frequency zero.

2. THE FILTER BANK

The analysis and synthesis filters, $h_k(n)$ and $f_k(n)$, respectively, of the considered cosine-modulated filter bank are given by

$$h_k(n) = \rho_k p_a(n) \cos\left(\frac{\pi k}{M}(n+0.5)\right), \tag{1}$$

$$f_k(n) = \rho_k p_s(n) \cos\left(\frac{\pi k}{M}(n+0.5-M)\right) \tag{2}$$

$$k = 0, \dots, M-1, \quad n = 0, \dots N-1$$

$$\rho_0 = \sqrt{2}, \quad \rho_k = 2 \text{ for } k \neq 0$$

where $p_a(n)$ and $p_s(n)$ denote the length-N analysis and synthesis prototype filters, respectively. The constraints for perfect reconstruction of the filter bank can be expressed by means of the analysis and synthesis polyphase matrix, $\mathbf{E}(z)$ and $\mathbf{R}(z)$, respectively, that write

$$\mathbf{E}(z) = \mathbf{C}_1 \begin{bmatrix} \mathbf{g}_0(z^2) \\ z^{-1}\mathbf{g}_1(z^2) \end{bmatrix}, \; \mathbf{R}(z) = \begin{bmatrix} z^{-1}\mathbf{k}_0(z^2) & \mathbf{k}_1(z^2) \end{bmatrix} \mathbf{C}_2^T \tag{3}$$

with

$$[\mathbf{C}_1]_{k,n} = \rho_k \cos(\frac{\pi k}{M}(n+0.5))$$
$$[\mathbf{C}_2]_{k,n} = \rho_k \cos(\frac{\pi k}{M}(n+0.5-M))$$
$$k = 0,\dots,M-1, \quad n = 0,\dots,2M-1$$
$$\mathbf{g}_0(z) = \text{diag}\{G_0(z),\dots,G_{M-1}(z)\}$$
$$\mathbf{g}_1(z) = \text{diag}\{G_M(z),\dots,G_{2M-1}(z)\}$$
$$\mathbf{k}_0(z) = \text{diag}\{K_{2M-1}(z),\dots,K_M(z)\}$$
$$\mathbf{k}_1(z) = \text{diag}\{K_{M-1}(z),\dots,K_0(z)\}$$

and $g_\ell(m) = p_a(2mM+\ell)$ and $k_\ell(m) = p_s(2mM+\ell)$ being the ℓ-th type-1 polyphase component of the analysis and synthesis prototype, respectively. Taking into consideration that $\mathbf{R}(z)\mathbf{E}(z) = z^{-2r}\mathbf{I}_M$ has to be satisfied for perfect reconstruction with an overall system delay of $N-1$ samples [9] and that the matrices $\mathbf{C}_2$ and $\mathbf{C}_1$ satisfy

$$\mathbf{C}_2^T\mathbf{C}_1 = 2M\begin{bmatrix}\mathbf{J}_M & \mathbf{I}_M\\ \mathbf{I}_M & \mathbf{J}_M\end{bmatrix}, \tag{4}$$

the constraints for PR can be expressed as

$$\mathbf{K}_\ell(z)\mathbf{G}_\ell(z) = \begin{bmatrix}\frac{z^{-2r}}{2M} & 0\\ 0 & \frac{z^{-2r}}{2M}\end{bmatrix},\ \ell = 0,\dots,M/2-1 \tag{5}$$

with

$$\mathbf{K}_\ell(z) = \begin{bmatrix}K_\ell(z^2) & z^{-1}K_{M+\ell}(z^2)\\ z^{-1}K_{2M-1-\ell}(z^2) & K_{M-1-\ell}(z^2)\end{bmatrix}$$
$$\mathbf{G}_\ell(z) = \begin{bmatrix}G_{M-1-\ell}(z^2) & z^{-1}G_{M+\ell}(z^2)\\ z^{-1}G_{2M-1-\ell}(z^2) & G_\ell(z^2)\end{bmatrix}$$

Given the analysis polyphase filters in (5) the matrix $\mathbf{K}_\ell(z)$ writes

$$\mathbf{K}_\ell(z) = \frac{z^{-2r}}{2M}\cdot \tag{6}$$
$$\frac{\begin{bmatrix}G_\ell(z^2) & -z^{-1}G_{M+\ell}(z^2)\\ -z^{-1}G_{2M-1-\ell}(z^2) & G_{M-1-\ell}(z^2)\end{bmatrix}}{G_\ell(z^2)G_{M-1-\ell}(z^2) - z^{-2}G_{M+\ell}(z^2)G_{2M-1-\ell}(z^2)}$$

where the denominator has to be a monomial in order to obtain FIR synthesis polyphase components. Thus, the PR constraint on the analysis filters writes

$$G_\ell(z)G_{M-1-\ell}(z) - z^{-1}G_{M+\ell}(z)G_{2M-1-\ell}(z) = \alpha_\ell^2 z^{-r} \tag{7}$$

and the synthesis polyphase filters can be obtained from the analysis ones as

$$K_\ell(z) = \frac{G_\ell(z)}{2M\alpha_\ell^2}, \quad K_{\ell+M}(z) = -\frac{G_{\ell+M}(z)}{2M\alpha_\ell^2} \tag{8}$$
$$\ell = 0,\dots,M-1, \quad \alpha_{2M-1-\ell} = \alpha_{M-1-\ell} = \alpha_\ell$$

3. LINEAR PHASE ANALYSIS AND SYNTHESIS FILTERS

It can easily be verified that the analysis and synthesis filters of length $N = 2rM + M$ in (1) and (2) are linear phase if the prototype filters $p_a(n)$ and $p_s(n)$ are linear phase. In this case, the following relationship holds true for the analysis polyphase filters

$$G_{M-1-\ell}(z) = z^{-r}\tilde{G}_\ell(z) = \frac{z^{-r}}{2M\alpha_\ell^2}\tilde{K}_\ell(z) \tag{9}$$

$$G_{2M-1-\ell}(z) = z^{-r+1}\tilde{G}_{\ell+M}(z) = -\frac{z^{-r+1}}{2M\alpha_\ell^2}\tilde{K}_{\ell+M}(z) \tag{10}$$
$$\ell = 0,\dots,M/2-1$$

In the upper two equations we have taken into account that the polyphase filters $G_0(z)$ to $G_{M-1}(z)$ are of length $r+1$ and the remaining ones of length r. Equation (7) now writes

$$G_\ell(z)\tilde{K}_\ell(z) + G_{M+\ell}(z)\tilde{K}_{M+\ell}(z) = \frac{1}{2M} \tag{11}$$

Note that this PR constraint differs from the one derived for DCT-IV filter banks in [8] just by the fact that it contains analysis and synthesis polyphase components and that due to (8) analysis and synthesis prototype filter cannot be equal.

4. THE LATTICE STRUCTURE

A lattice structure that automatically keeps the linear phase property of the analysis prototype filter and satisfies (7) is given by the following formulation:

$$\begin{bmatrix}G_\ell(z)\\ G_{\ell+M}(z)\end{bmatrix} = \alpha_\ell\begin{bmatrix}z^{-1} & 0\\ 0 & -1\end{bmatrix}\cdot \tag{12}$$
$$\prod_{i=1}^{r-1}\left(\begin{bmatrix}\cosh(\theta_{\ell,i}) & \sinh(\theta_{\ell,i})\\ \sinh(\theta_{\ell,i}) & \cosh(\theta_{\ell,i})\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\right)\begin{bmatrix}\cosh(\theta_{\ell,0})\\ \sinh(\theta_{\ell,0})\end{bmatrix}$$

$$\begin{bmatrix}G_{M-1-\ell}(z)\\ G_{2M-1-\ell}(z)\end{bmatrix} = \alpha_\ell\cdot \tag{13}$$
$$\prod_{i=1}^{r-1}\left(\begin{bmatrix}\cosh(\theta_{\ell,i}) & -\sinh(\theta_{\ell,i})\\ -\sinh(\theta_{\ell,i}) & \cosh(\theta_{\ell,i})\end{bmatrix}\begin{bmatrix}z^{-1} & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\cosh(\theta_{\ell,0})\\ -\sinh(\theta_{\ell,0})\end{bmatrix}$$

A second solution is given when swapping the left-hand sides of the upper equations. That is, swapping $G_\ell(z)$ with $G_{M-1-\ell}(z)$ as well as $G_{\ell+M}(z)$ with $G_{2M-1-\ell}(z)$. For the synthesis filters we can use the same structure with a different scaling factor. The new scaling factor can be calculated using (8).

It can easily be verified that the upper lattice structure satisfies the PR constraint, when reformulating (7) as

$$\begin{bmatrix}G_\ell(z) & G_{\ell+M}(z)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -z^{-1}\end{bmatrix}\begin{bmatrix}G_{M-1-\ell}(z)\\ G_{2M-1-\ell}(z)\end{bmatrix} = \alpha_\ell^2 z^{-r} \tag{14}$$

5. NO DC LEAKAGE

The filter bank is free of DC leakage if the analysis filters $H_k(\omega)$ for $k > 0$, have at least one zero at frequency zero, i.e. $H_k(0) = 0$ for $k > 0$. The lowpass filter has to satisfy $H_0(0) = \beta$. The vector $\mathbf{h}(z) = [H_0(z), \ldots, H_{M-1}(z)]^T$ containing all analysis filters can be written as [9]

$$\mathbf{h}(z) = \mathbf{E}(z^M)\mathbf{e}(z) \tag{15}$$

with $\mathbf{E}(z)$ from (3) and $\mathbf{e}(z) = [1, z^{-1}, \ldots, z^{-(M-1)}]^T$ being the delay chain. To obtain a formulation for a DCT-II filter bank without DC leakage, we replace z in (15) by $e^{j\omega}$ and set $\omega = 0$. Thus, $\mathbf{h}(1)$ writes

$$\mathbf{h}(1) = \mathbf{C}_1 \begin{bmatrix} \sum_i g_0(i) \\ \vdots \\ \sum_i g_{2M-1}(i) \end{bmatrix} \tag{16}$$

and has to satisfy $\mathbf{h}(1) = [\beta, 0, \ldots, 0]^T$. A solution is

$$\sum_{i=0}^{r} g_\ell(i) + \sum_{i=0}^{r-1} g_{2M-1-\ell}(i) = \frac{\beta}{M\sqrt{2}}, \quad \ell = 0, \ldots, M-1 \tag{17}$$

This is the condition the filter bank has to satisfy in order to provide no DC leakage. Note that the result is in accordance with the fact that the prototype filters should be linear phase. Since the polyphase filters $G_\ell(z)$ and $G_{M-1-\ell}(z)$ as well $G_{\ell+M}(z)$ and $G_{2M-1-\ell}(z)$ contain the same coefficients, but in inverse order, both sums, i.e. for ℓ and $\ell + M$ are equal.

The lattice structure presented in the last section can be used such as to guarantee not only PR but also no DC leakage. We first have to replace z by one in (12) and (13) obtaining

$$\begin{bmatrix} \sum_i g_\ell(i) \\ \sum_i g_{\ell+M}(i) \end{bmatrix} = \alpha_\ell \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \prod_{i=1}^{r-1} \begin{bmatrix} \cosh(\theta_{\ell,i}) & \sinh(\theta_{\ell,i}) \\ \sinh(\theta_{\ell,i}) & \cosh(\theta_{\ell,i}) \end{bmatrix} \begin{bmatrix} \cosh(\theta_{\ell,0}) \\ \sinh(\theta_{\ell,0}) \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} \sum_i g_{M-1-\ell}(i) \\ \sum_i g_{2M-1-\ell}(i) \end{bmatrix} = \alpha_\ell \cdot \prod_{i=1}^{r-1} \begin{bmatrix} \cosh(\theta_{\ell,i}) & -\sinh(\theta_{\ell,i}) \\ -\sinh(\theta_{\ell,i}) & \cosh(\theta_{\ell,i}) \end{bmatrix} \begin{bmatrix} \cosh(\theta_{\ell,0}) \\ -\sinh(\theta_{\ell,0}) \end{bmatrix} \tag{19}$$

Using basic theorems of the hyperbolic sine and cosine function, the upper relationships can be formulated as

$$\begin{bmatrix} \sum_i g_\ell(i) \\ \sum_i g_{\ell+M}(i) \end{bmatrix} = \begin{bmatrix} \sum_i g_{M-1-\ell}(i) \\ \sum_i g_{2M-1-\ell}(i) \end{bmatrix} = \alpha_\ell \begin{bmatrix} \cosh(\sum_{i=0}^{r-1} \theta_{\ell,i}) \\ -\sinh(\sum_{i=0}^{r-1} \theta_{\ell,i}) \end{bmatrix} \tag{20}$$

and the constraint for no DC leakage given in (17) is fulfilled if

$$\cosh(\sum_{i=0}^{r-1} \theta_{\ell,i}) - \sinh(\sum_{i=0}^{r-1} \theta_{\ell,i}) = \frac{\beta}{\alpha_\ell M\sqrt{2}} \tag{21}$$

Using $\cosh(x) = \sqrt{1 + \sinh^2(x)}$ as well as $\text{arsinh}(y) = \ln(y + \sqrt{y^2 + 1})$ we obtain as a constraint on the angles

$$\sum_{i=0}^{r-1} \theta_{\ell,i} = \ln(\frac{\alpha_\ell M\sqrt{2}}{\beta}) \tag{22}$$

Thus, if we remove one angle from the optimization parameter space and use that angle in order to satisfy the upper equation we can easily guarantee that the filter bank has no DC leakage.

6. PROTOTYPE DESIGN RESULTS

For the prototype design we used a nonlinear optimization routine (fminu.m from MATLAB) and as a cost function the sum of the stopband energies of the analysis and synthesis prototypes. The angles $\theta_{\ell,i}$ with $i = 0, \ldots, r-2$ where optimized such as to minimize the cost function. For each ℓ we tried both realization possibilities for the polyphase filters: the one given in (12) and (13) as well as the one where the $G_\ell(z)$ and $G_{M-1-\ell}(z)$ were swapped as well as $G_{\ell+M}(z)$ and $G_{2M-1-\ell}(z)$. Out of all possibilities we chose the one with the lowest cost function. Figure 1 shows the amplitude response of the analysis and synthesis prototype filters for $N = 28$ and $M = 4$. The value β has been chosen as $\sqrt{M}$ in order to obtain the same value of the amplitude response at frequency zero for the analysis and synthesis prototype.

In a second example we chose the same filter parameters but weighted the stopband energy of the analysis prototype with a factor of 100 before adding it to the synthesis prototype's stopband energy, thus giving more emphasis on the optimization of the analysis prototype than the synthesis prototype. The resulting prototype filters are shown in Figure 2. Figure 3 shows the analysis filters using the prototype in Figure 2. It can be seen that all analysis filters apart from the lowpass filter have a zero at $\omega = 0$.

7. CONCLUSION

In this paper we have presented a class of modulated filter banks based on a DCT-II with linear-phase analysis and synthesis filters and no DC leakage. Although we have only treated the case where the filter length is given as $N = 2rM + M$ and M even, the formalism can easily be extended to odd M and arbitrary filter lengths using similar derivations as in [6, 5]. In the simulations it turns out that a

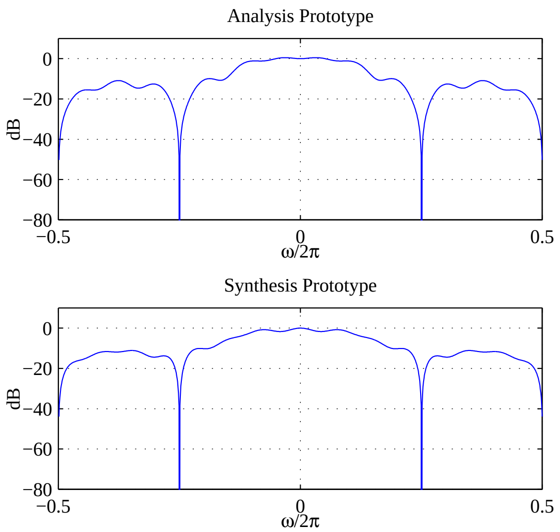

Figure 1: Normalized amplitude response of analysis and synthesis prototype for $N = 28$ and $M = 4$

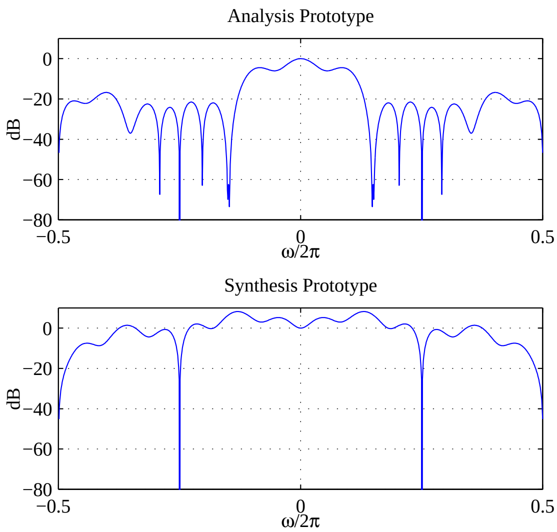

Figure 2: Normalized amplitude response of analysis and synthesis prototype for $N = 28$ and $M = 4$ when putting a 100 times higher weight on the optimization of the analysis prototype

crucial point of this filter bank is the fact that analysis and synthesis prototypes are not equal but strictly connected via (8). It has not been investigated yet, whether this connection prevents the design of prototypes with better properties than the ones shown in the examples, or if the optimization routine stopped in a local minimum.

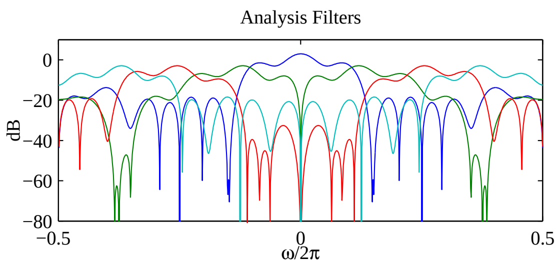

Figure 3: Analysis filters for the prototype in Figure 2

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