

## Shift Variance and Cyclostationarity in Multirate Filter Banks

Til Aach

Institute for Signal Processing, University of Lübeck  
 Ratzeburger Allee 160, D-23538 Lübeck, GERMANY  
 email: aach@isip.uni-luebeck.de

### ABSTRACT

Multirate filtering generally introduces time-varying phenomena into the decomposed signals. These phenomena depend on whether the signals are viewed as deterministic, or as wide-sense (WS) stationary random signals. We consider the behaviour of subband energy spectra and subband energies for deterministic signals, and of subband power spectra and subband power for WS stationary random signals. We show that aliasing in decimation causes energy spectra and energies of deterministic signals to be translation variant, but leaves WS stationary random signals WS stationary. Interpolation by upsampling and anti-imaging filtering attenuates translation variance of deterministic signals, but converts WS stationary random signals into WS cyclostationary random signals. In perfect reconstruction (PR) filter banks, the time-varying effects of the filter bank channels cancel out. Subband signal processing such as filtering or quantization, however, distorts the balance between the filter channels and lets these effects appear in the output signal. Based on our analysis, we therefore derive quantitative measures for translation variance and cyclostationarity. These may be used to compare different multirate filter banks.

### 1. Introduction

Conversion of sampling rates in critically sampled filter banks generally causes the subband coefficients to be translation variant. For instance, when a deterministic input signal is shifted along time axis or spatial coordinates, its subband coefficients will not translate in the same manner, and subband energy may change significantly [1]. Adaptive subband filtering, e.g. for multirate noise reduction and enhancement of X-ray images [2], may therefore generate translation-dependent results. This is particularly annoying in interframe image sequence processing and compression: shift variance makes motion estimation more difficult [3], and makes the processing results for given objects to vary as the objects move. Approaches to reduce or avoid shift variance include to operate without sampling rate conversion [4], cycle spinning [5], and signal normalization [6, 7].

If the signals are modelled as wide-sense (WS) stationary random signals, they are described by statistical properties which, by definition, do not depend on time shifts. Analysis of shift variance is therefore not applicable to stationary random signals. However, a WS stationary random signal does generally not remain WS stationary when passing through a filter bank. While decimation changes the power spectrum, but leaves the signal WS stationary, interpolation causes correlations and power spectra to become periodically time dependent, or cyclostationary [8, 9]. How severe shift variance and cyclostationarity are, depends on the anti-aliasing filter before downsampling, and the anti-imaging filter after

upsampling. In perfect reconstruction (PR) filter banks, PR constraints impose close relations between these filters.

An earlier approach to quantify aliasing in multirate filter banks is the so-called non-aliasing energy ratio (NER) [10]. However, while complementing the energy compaction measures used in transform coding [11], the NER makes no distinction between deterministic and WS stationary random signals. Also, it implicitly averages over the cyclostationarity, and does not separate between effects caused by decimation and by interpolation. Here, we therefore develop the effects energy and power spectra undergo in multirate filter channels in a parallel, comparative way. From this analysis, we derive quantitative measures for shift variance and cyclostationarity.

The multirate filter bank is shown in Fig. 1: In each channel, the input signal  $s(n)$  is first filtered by an analysis (or anti-aliasing) filter  $H_i(z)$ ,  $i = 0, \dots, M$ , yielding  $T_i(z) = S(z)H_i(z)$ . It is then downsampled by a factor  $M$  to  $x_i(n) = t_i(Mn)$ . Signal synthesis starts by upsampling according to

$$v_i(n) = \begin{cases} x_i(\frac{n}{M}) & \text{for } \frac{n}{M} \text{ integer} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The upsampled signal is filtered by the synthesis filter  $G_i(z)$  to attenuate or remove frequency images (i.e., the replicas of the spectrum  $X_i(e^{j\omega})$  lying within  $-\pi < \omega \leq \pi$ ) by interpolating between the samples  $x_i(n/M)$  [10, 12].

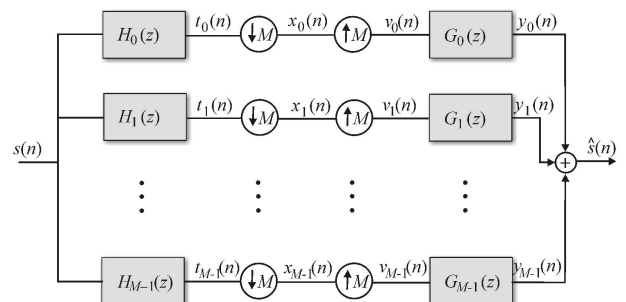


Figure 1:  $M$ -channel critically sampled multirate filter bank.

### 2. Deterministic Signals

The energy spectrum  $R_{ss}^E(e^{j\omega})$  of a deterministic real  $L_2$ -signal  $s(n) \circ \bullet S(e^{j\omega})$  is given as the Fourier transform of the autocorrelation sequence (ACS) according to

$$r_{ss}^E(n) = s(-n) * s(n) \circ \bullet R_{ss}^E(e^{j\omega}) = |S(e^{j\omega})|^2 \quad (2)$$

The (finite) energy  $E_s$  of  $s(n)$  can be calculated along the time or frequency axis by

$$E_s = r_{ss}^E(0) = \sum_{n=-\infty}^{\infty} s^2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(e^{j\omega})|^2 d\omega \quad (3)$$

## 2.1 Decimation

The spectrum of the downsampled signal  $x_i(n)$  is  $X_i(z) = \frac{1}{M} \sum_{k=0}^{M-1} T_i(z^{1/M} W^k)$ , with  $W = e^{-j\frac{2\pi}{M}}$ . The correlation sequence  $r_{x_i x_j}^E(n)$  and the (cross) energy spectrum  $R_{x_i x_j}^E(z)$  of  $x_i(n)$  and  $x_j(n)$  are

$$r_{x_i x_j}^E(n) = x_i(-n) * x_j(n) \circ \bullet R_{x_i x_j}^E(z) = X_i(z^{-1}) X_j(z) \quad (4)$$

yielding for the energy spectrum

$$R_{x_i x_j}^E(z) = \frac{1}{M^2} \sum_{k=0}^{M-1} T_j(z^{\frac{1}{M}} W^k) \cdot \sum_{l=0}^{M-1} T_i(z^{-\frac{1}{M}} W^l) \quad (5)$$

We now regard a shift of  $s(n)$  by  $m$  samples to  $s(n-m)$ . In a uniform  $M$ -channel critically sampled filter bank, shift-induced effects will be periodic with period  $M$ . It therefore suffices to consider  $m$  for  $m = 0, \dots, M-1$ . Inserting the phase factor  $z^{-m}$  in both sums of Eq. (5) yields the shift-variant energy spectra

$$R_{x_i x_j}^E(m, z) = \frac{1}{M^2} \sum_{k=0}^{M-1} T_j(z^{\frac{1}{M}} W^k) \cdot W^{-km} \cdot \sum_{l=0}^{M-1} T_i(z^{-\frac{1}{M}} W^l) \cdot W^{-lm} \quad (6)$$

This expression can be regarded as the product of the discrete Fourier spectra of  $J(k) = T_j(z^{\frac{1}{M}} W^k)$  and  $I(l) = T_i(z^{-\frac{1}{M}} W^l)$ . Rewriting this product as the discrete Fourier spectrum of the convolution of  $J(k)$  and  $I(k)$ , and grouping the energy spectra for  $m = 1, \dots, M-1$  into a vector, we obtain

$$\begin{bmatrix} R_{x_i x_j}^E(0, z), \dots, R_{x_i x_j}^E(M-1, z) \end{bmatrix}^T = \frac{\mathbf{W}}{M^2} \begin{bmatrix} A_0^{ij}(z^{1/M}), \dots, A_{M-1}^{ij}(z^{1/M}) \end{bmatrix}^T \quad (7)$$

where  $\mathbf{W}$  is the Fourier matrix, and  $A_k^{ij}(z)$  is the convolution of modulated spectra

$$A_k^{ij}(z) = \sum_{l=0}^{M-1} T_j(z W^{-l}) T_i(z^{-1} W^{l-k}) \quad (8)$$

Focussing now on  $i = j$ , i.e., excluding cross energy spectra,  $R_{x_i x_i}^E(m, z)$  is independent of  $m$ , or shift invariant, for any  $S(z)$  if and only if  $A_k^{ii}(z^{1/M}) = 0, k = 1, \dots, M$ . This is identical to the absence of aliasing. The shift-independent subband energy spectrum is then equal to the average

$$R_{x_i x_i}^E(z) = \bar{R}_{x_i x_i}^E(z) = \frac{1}{M^2} A_0^{ii}(z^{1/M}) \quad (9)$$

In practice, FIR filters will not prevent aliasing. The deviations  $\Delta R_{x_i x_i}^E(m, z) = R_{x_i x_i}^E(m, z) - \bar{R}_{x_i x_i}^E(z)$  from the average spectrum are then given by non-DC components of Eq. (7):

$$\begin{bmatrix} \Delta R_{x_i x_i}^E(0, z), \dots, \Delta R_{x_i x_i}^E(M-1, z) \end{bmatrix}^T = \frac{\mathbf{W}}{M^2} \begin{bmatrix} 0, A_1^{ii}(z^{1/M}), \dots, A_{M-1}^{ii}(z^{1/M}) \end{bmatrix} \quad (10)$$

With the energy  $e_k^x(i)$

$$e_k^x(i) = \frac{1}{2\pi M^2} \int_{-\pi}^{\pi} A_k^{ii}(e^{j\frac{\omega}{M}}) d\omega \quad (11)$$

where  $A_k^{ii}(e^{j\frac{\omega}{M}})$  is the convolution

$$A_k^{ii}(e^{j\frac{\omega}{M}}) = \sum_{l=0}^{M-1} T_i(e^{j\frac{\omega+2\pi l}{M}}) T_i^*(e^{j\frac{\omega+2\pi(l-k)}{M}}) \quad (12)$$

the energy average is  $\bar{E}_{x_i x_i} = e_0^x(i)$ . The shift-induced energy deviations from the average are

$$[\Delta E_{x_i x_i}(0), \dots, \Delta E_{x_i x_i}(M-1)]^T = \mathbf{W} [0, e_1^x(i), \dots, e_{M-1}^x(i)]^T \quad (13)$$

As a quantitative measure of the susceptibility of the subband energy to shifts, we define the normalized mean square deviation from the average energy as

$$C_e^2(x_i) = \frac{\frac{1}{M} \sum_{m=0}^{M-1} (\Delta E_{x_i x_i}(m))^2}{(\bar{E}_{x_i x_i})^2} = \frac{M \sum_{k=1}^{M-1} |e_k^x(i)|^2}{(e_0^x(i))^2} \quad (14)$$

For a given input spectrum  $S(z)$ , this measure depends only on the properties of the analysis filters  $H_i(z)$ .

## 2.2 Interpolation

Upsampling stretches input signals  $x_i(n)$ , and correspondingly compresses energy spectra to  $R_{v_i v_j}^E(z) = R_{x_i x_j}^E(z^M)$  [13]. Auto- and cross correlation sequences are therefore upsampled like the signals themselves. After filtering by the synthesis filters  $G_i(z)$ , the energy spectra obey

$$R_{v_i v_j}^E(z) = G_i(z^{-1}) R_{x_i x_j}^E(z^M) G_j(z) \quad (15)$$

Unlike decimation, interpolation causes no shift dependencies of energy spectra and energies.

## 2.3 Decimation and Interpolation

We now examine the energy-related shift dependencies of the output signals  $y_i(n)$  of the filter bank channels in Fig. 1. From decimation, the input energy spectra are shift-dependent. Eq. (15) therefore becomes

$$R_{v_i v_j}^E(m, z) = G_i(z^{-1}) R_{x_i x_j}^E(m, z^M) G_j(z) \quad (16)$$

Inserting Eq. (7), we obtain for the output energy spectra

$$\begin{bmatrix} R_{y_i y_j}^E(0, z), \dots, R_{y_i y_j}^E(M-1, z) \end{bmatrix}^T = \frac{\mathbf{W}}{M^2} \begin{bmatrix} B_0^{ij}(z), \dots, B_{M-1}^{ij}(z) \end{bmatrix}^T \quad (17)$$

where

$$B_k^{ij}(z) = G_i(z^{-1}) A_k^{ij}(z) G_j(z) \quad (18)$$

For  $i = j$ , the average energy spectrum of  $y_i(n)$  is

$$\bar{R}_{y_i y_i}^E(z) = \frac{1}{M^2} B_0^{ii}(z) \quad (19)$$

and the remaining deviations after synthesis filtering are

$$\begin{bmatrix} \Delta R_{y_i y_i}^E(0, z), \dots, \Delta R_{y_i y_i}^E(M-1, z) \end{bmatrix}^T = \frac{\mathbf{W}}{M^2} \begin{bmatrix} 0, B_1^{ii}(z), \dots, B_{M-1}^{ii}(z) \end{bmatrix} \quad (20)$$

For the output signals  $y_i(n)$ , a measure of shift dependence can be derived analog to Eq. (14): with the energy

$$e_k^y(i) = \frac{1}{2\pi M^2} \int_{-\pi}^{\pi} B_k^{ii}(e^{j\omega}) d\omega \quad (21)$$

where

$$B_k^{ii}(e^{j\omega}) = \sum_{l=0}^{M-1} T_i(e^{j(\omega + \frac{2\pi l}{M})}) T_i^*(e^{j(\omega + \frac{2\pi(l-k)}{M})}) |G_i(e^{j\omega})|^2 \quad (22)$$

the energy average is  $\bar{E}_{y_i y_i} = e_0^y(i)$ . The energy deviations are

$$[\Delta E_{y_i y_i}(0), \dots, \Delta E_{y_i y_i}(M-1)]^T = \mathbf{W} [0, e_1^y(i), \dots, e_{M-1}^y(i)]^T \quad (23)$$

The normalized mean square energy deviation of the output signal energies therefore is

$$C_e^2(y_i) = \frac{\frac{1}{M} \sum_{m=0}^{M-1} (\Delta E_{y_i y_i}(m))^2}{(\bar{E}_{y_i y_i})^2} = \frac{M \sum_{k=1}^{M-1} |e_k^y(i)|^2}{(e_0^y(i))^2} \quad (24)$$

Evidently, the synthesis filters tend to attenuate the shift dependencies. If the bandwidths of  $G_i(z)$  are sufficiently narrow (less than  $2\pi/M$  if the analysis filters  $H_i(z)$  do not prevent aliasing) they could even eliminate the shift variances in the output signals. In perfect reconstruction (PR) filter banks, analysis and synthesis filters are, however, closely related. For instance, in QMF banks, filters are designed from a common lowpass prototype  $H(z)$  by modulation, coefficient reversal and shifts. Hence,  $|H_i(e^{j\omega})| = |G_i(e^{j\omega})|$ . Shift dependencies can then not be filtered out.

### 3. WS Stationary Random Signals

The ACS  $r_{ss}(n)$  of a WS stationary signal  $s(n)$  is the expectation

$$r_{ss}(n) = E[s(m)s(m+n)] = \int \int_{-\infty}^{\infty} xyp_{ss}(x,y,n) dx dy \quad (25)$$

where  $p_{ss}(x,y,n)$  is the joint probability density function of any two samples  $s(m)$  and  $s(m+n)$ . The power spectrum  $R_{ss}(e^{j\omega})$  is the Fourier transform of  $r_{ss}(n)$ . The power of  $s(n)$  is

$$P_s = E[s^2(m)] = r_{ss}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_{ss}(e^{j\omega}) d\omega \quad (26)$$

The expectations are ensemble averages taken over the same  $m, n$  of different realizations of the random signal  $s(n)$ .

#### 3.1 Decimation

The power spectrum of the filtered signal  $t_i(n)$  is

$$R_{t_i t_i}(z) = H_i(z^{-1}) R_{ss}(z) H_i(z) \quad (27)$$

The ACS of the downsampled signal  $x_i(n)$  is

$$\begin{aligned} r_{x_i x_i}(n) &= E[t_i(Mm)t_i(M(m+n))] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyp_{t_i t_i}(x,y,Mn) dx dy \\ &= r_{t_i t_i}(Mn) \end{aligned} \quad (28)$$

The ACS is downsampled like the signal itself. The power spectrum therefore undergoes the same changes as a deterministic signal's spectrum, yielding

$$R_{x_i x_i}(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_i(z^{-1/M} W^{-k}) R_{ss}(z^{1/M} W^k) H_i(z^{1/M} W^k) \quad (29)$$

With a  $M \times M$  diagonal matrix of modulated versions of  $R_{ss}(z)$  defined as

$$\mathbf{R}_{ss}(z) = \text{diag}[R_{ss}(z), R_{ss}(zW), \dots, R_{ss}(zW^{M-1})] \quad (30)$$

and the modulation vector  $\mathbf{h}_{im}(z)$  of the  $i$ -th filter  $H_i(z)$

$$\mathbf{h}_{im}(z) = [H_i(z), H_i(zW), \dots, H_i(zW^{M-1})]^T \quad (31)$$

Eq. (29) becomes

$$R_{x_i x_i}(z) = \frac{1}{M} \mathbf{h}_{im}^T(z^{-1/M}) \mathbf{J}_M \mathbf{R}_{ss}(z^{1/M}) \mathbf{h}_{im}(z^{1/M}) \quad (32)$$

$\mathbf{J}_M$  is a  $M \times M$ -matrix permutation matrix, which reverses the order of vector entries in  $\mathbf{h}_{im}^T(z^{-1/M})$  except the first one, thus accounting for the  $W^{-k} = W^{M-k}$ -dependence in Eq. (29). While  $R_{x_i x_i}(z)$  generally contains aliasing, the subband signal  $x_i(n)$  is still WS stationary. Its power, variance and mean value are the same as those of  $t_i(n)$ . Aliasing only changes the shape of  $R_{x_i x_i}(e^{j\omega})$  by increasing the contributions from higher frequencies.

#### 3.2 Interpolation

Upsampling inserts  $M-1$  deterministic zeros between two successive samples of the input signal  $x_i(n)$ . The upsampled signal is therefore not WS stationary [13]. Its ACS and power spectrum vary periodically along the time coordinate  $m$ . Interpolation by the anti-imaging filter  $G_i(z)$  smoothes over these periodic variations, but will in practice not eliminate these. Output ACS  $r_{y_i y_i}(m, n)$  and output power spectrum  $R_{y_i y_i}(m, z)$  therefore depend periodically on  $m$ ,  $m = 0, \dots, M-1$ , making  $y_i(n)$  WS cyclostationary. To derive the relation between the correlation structure of  $x_i(n)$  and  $y_i(n)$ , we decompose the interpolator into its polyphase representation in Fig. 2. Evidently, the output ACS  $r_{y_i y_i}(m, n)$  can then be rewritten as the cross correlation of two polyphase filter output signals [13]. Grouping the  $M$  different power spectra  $R_{y_i y_i}(m, z)$  into a vector, we derived in [13] that this vector is given by

$$[R_{y_i y_i}(0, z), \dots, R_{y_i y_i}(M-1, z)]^T = \frac{R_{x_i x_i}(z^M) \mathbf{W} G_i(z)}{M} \mathbf{g}_{im}(z^{-1}) \quad (33)$$

where  $\mathbf{g}_{im}(z)$  is the modulation vector of  $G_i(z)$ . Clearly, for any WS stationary  $x_i(n)$ ,  $y_i(n)$  is WS stationary if and only if

$$G_i(z) \cdot \mathbf{g}_{im}(z^{-1}) = [G_i(z) G_i(z^{-1}), 0, \dots, 0]^T \quad (34)$$

i.e. the bandwidth of  $G_i(z)$  must be  $2\pi/M$  or less. In this ideal case, the power spectra  $R_{y_i y_i}(m, z)$  are equal to the average

$$\frac{1}{M} \sum_{m=0}^{M-1} R_{y_i y_i}(m, z) = \frac{R_{x_i x_i}(z^M)}{M} G_i(z) G_i(z^{-1}) \quad (35)$$

If condition (34) is not met, the deviations  $\Delta R_{y_i y_i}(m, z)$  from

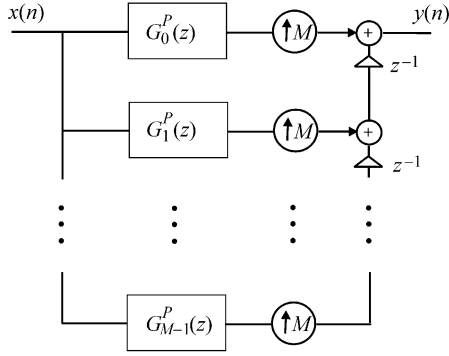


Figure 2: Polyphase representation of interpolation.  $G_k^p(z)$ ,  $k = 0, \dots, M-1$  denote the polyphase components of the anti-imaging filter  $G(z)$ .

the average are

$$\frac{R_{x_i x_i}(z^M)}{M} \mathbf{W} \mathbf{G}_i(z) [0, G_i(z^{-1}W), \dots, G_i(z^{-1}W^{M-1})]^T \quad (36)$$

Similar to the analysis of shift-dependent energy for deterministic signals, we find the periodically varying power  $P_{y_i}(m)$  of  $y_i(n)$  to

$$P_{y_i}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_{y_i y_i}(m, e^{j\omega}) d\omega, \quad m = 0, \dots, M-1 \quad (37)$$

With the power  $p_k^i$  from the overlap of  $G_i(z)$  and  $G_i(z^{-1}W^k)$

$$p_k^i = \frac{1}{2\pi M} \int_{-\pi}^{\pi} R_{x_i x_i}(e^{j\omega M}) G_i(e^{j\omega}) G_i(e^{-j(\omega+2\pi k/M)}) d\omega \quad (38)$$

and inserting Eq. (33), the vector of output powers is

$$[P_{y_i}(0), \dots, P_{y_i}(M-1)]^T = \mathbf{W} [p_0^i, \dots, p_{M-1}^i]^T \quad (39)$$

The average power  $\bar{P}_{y_i}$  of  $y_i(n)$  then is

$$\bar{P}_{y_i} = \frac{1}{M} \sum_{k=0}^{M-1} P_{y_i}(k) = p_0^i \quad (40)$$

and the power ripple  $\Delta P_{y_i}(m)$  is

$$[\Delta P_{y_i}(0), \dots, \Delta P_{y_i}(M-1)]^T = \mathbf{W} [0, p_1^i, \dots, p_{M-1}^i]^T \quad (41)$$

As in Eq. (14) for energies, we can quantify the generation of cyclostationarity by the normalized mean square power deviation from the average power as

$$C_p^2(y_i) = \frac{\frac{1}{M} \sum_{m=0}^{M-1} (\Delta P_{y_i}(m))^2}{\left(\frac{1}{M} \sum_{m=0}^{M-1} P_{y_i}(m)\right)^2} = \frac{M \sum_{k=1}^{M-1} |p_k^i|^2}{(p_0^i)^2} \quad (42)$$

For a given input power spectrum  $R_{x_i x_i}(z)$ , this measure depends only on the properties of the synthesis filters  $G_i(z)$ .

### 3.3 Decimation and Interpolation

To calculate the relation between the vector with output power spectra  $R_{y_i y_i}(m, z)$  and the input power spectrum  $R_{ss}(z)$ , we cascade Eqs. (32) and (33), what leads to

$$[R_{y_i y_i}(0, z), R_{y_i y_i}(1, z), \dots, R_{y_i y_i}(M-1, z)]^T = \frac{1}{M^2} \mathbf{h}_{im}^T(z^{-1}) \mathbf{J}_M \mathbf{R}_{ss}(z) \mathbf{h}_{im}(z) \mathbf{G}_i(z) \mathbf{W} \mathbf{g}_{im}(z^{-1}) \quad (43)$$

Modifying the cyclostationarity measure  $C_p^2(y_i)$  in Eq. (42) to cover the full filter bank is straightforward: in Eq. (38),  $R_{x_i x_i}(e^{j\omega M})$  becomes with Eq. (29)

$$R_{x_i x_i}(e^{j\omega M}) = \frac{1}{M} \sum_{k=0}^{M-1} R_{ss}(e^{j(\omega - \frac{2\pi k}{M})}) |H_i(e^{j(\omega - \frac{2\pi k}{M})})|^2 \quad (44)$$

$C_p^2(y_i)$  now depends on both analysis and synthesis filters, which, in PR filter banks, cannot be designed independently.

## 4. Conclusions

We have seen that decimation causes periodic shift dependence of the subband energy of deterministic signals, while interpolation causes WS stationary random signals to become periodically WS stationary, or WS cyclostationary. To quantify these effects, we have developed quantitative measures, which complement measures like energy compaction and decorrelation used to characterize transforms and subband decompositions. These measures can be evaluated analytically for given signals, such as AR(1)-processes, allowing individual assessments of filter bank channels as well as of the entire filter bank. We intend to evaluate different decomposition techniques, such as DCT, lapped transforms, and various multirate filter bank types, in the near future.

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