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Efficient biorthogonal cosine-modulated filter banks

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Abstract

Biorthogonal modulated filter banks, when compared to paraunitary ones, provide the advantage that the overall system delay can be chosen independently of the filter length, thus allowing to design low delay filter banks. They have recently been studied by several authors. In this paper, we connect two different design methods, namely the quadratic constrained least-squares optimization and the principle of cascading sparse self-inverse matrices. Moreover, we show how factorizations into zero-delay and maximum-delay matrices can be utilized in order to achieve desirable features such as structure-inherent perfect reconstruction, no DC leakage of the filter bank, and a low implementation cost. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Biorthogonal cosine-modulated filter banks; Low-delay filter banks; Perfect reconstruction; Efficient filter bank realization

1. Introduction

Modulated filter banks have been studied extensively in the literature within the last 10 years. They have shown to provide a very efficient implementation based on a prototype filter and a fast transform. The most popular modulation scheme is cosine modulation. However, other modulation schemes based on the discrete Fourier transform (DFT) also exist. Historically, the first modulated filter banks with perfect reconstruction were designed such as to be paraunitary [2,5,9–11]. In this special case, the impulse responses of the synthesis filters are flipped versions of the analysis ones and all filters are derived from one common prototype. However, since the overall system delay of such filter banks is directly related to the filter length, the desired features of a high stopband attenuation and a short overall system delay are contradictory. This problem has partly been overcome with the design of low-delay biorthogonal filter banks, where the delay can be chosen independently (within some fundamental limits) of the filter length and the number of subbands. In this class of filter banks, the synthesis filters are no longer flipped versions of the analysis filters—the analysis and synthesis filters may even be derived from different prototypes.

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Two principal approaches can be stated for the design of biorthogonal cosine-modulated filter banks with perfect reconstruction. The approach by Schuller et al. [12–15] uses filter bank realizations that structurally guarantee perfect reconstruction for arbitrary system delays. It is mainly based on a factorization of the analysis polyphase matrix into a transform and special sparse matrices which are easy to invert. The inverse matrices, which are also sparse, are then used on the synthesis side. Completeness of the factorization has been shown for all contiguous prototype filters, i.e. prototype filters that do not have any zero taps within their region of support. Prototype filters with desired features such as a high stopband attenuation are found by optimizing the free design parameters based on unconstrained nonlinear optimization methods.

The second approach is due to Nguyen et al., who derived explicit perfect reconstruction (PR) constraints for the polyphase components of the prototype filters [1,6,8]. The filter design is carried out by a quadratic-constrained least-squares (QCLS) optimization algorithm using the stopband energy of the prototypes' frequency responses as a cost function and the PR conditions as constraints. In [1] it has been shown that the same PR constraints also hold true for the DCT-II modulation scheme proposed in [3].

Both approaches provide certain advantages. Since in the latter case, the PR constraints are directly formulated, it can easily be verified whether or not a given pair of prototype filters yields perfect reconstruction. Necessary relations between the analysis and synthesis prototypes are also stated [1]. Furthermore, it gives good insight into the properties of PR prototypes. For example, based on this approach it can be shown that for certain combinations of filter lengths and overall system delay, some polyphase filters can only have one non-zero coefficient. This special case is not treated in the factorization proposed in [12–14]. However, the factorization approach from Schuller et al. offers many advantages concerning the implementation of the filter bank. First of all, the structure automatically guarantees PR. This also holds true when using integer-valued coefficients, because the same factorization coefficients are used on the analysis and synthesis side. Even for coefficients with infinite precision, it turns out that the implementation cost is nearly halved when compared to the direct realization of the polyphase filters as assumed by Nguyen et al. Another advantage of the approach is that it can be extended to time-varying filter banks [13] without much effort. Which of the two optimization methods (unconstrained and non-linear or constrained and quadratic) results in better filter designs, highly depends on the chosen optimization procedures and the complexity of the problem (i.e. filter length, number of subbands, etc.).

In this paper, we connect both approaches and show that all PR cosine-modulated filter banks with the modulation scheme considered by Nguyen et al. can be realized in a factorized form which shares similarities with the method proposed by Schuller et al. The factorization is derived directly from the PR constraints and also treats the general case where some polyphase filters contain coefficients being equal to zero, which is not covered by the approach in [12–14]. Instead of dealing with size $M \times M$ matrices as in [12–15], where M denotes the number of subbands of the filter bank, we just have to deal with size 2×2 matrices and realize $\lfloor M/2 \rfloor$ of them in parallel. Using this factorization, we show that the implementation cost can be significantly reduced (compared to a direct implementation of the polyphase filters). Furthermore, we show how to include certain useful features in the implementation. Such features can be the use of equal prototypes for analysis and synthesis, prototype filters with specified zeros at certain frequencies in order to yield filter banks without DC leakage, and integer coefficient prototypes.

The outline of the paper is as follows. After providing some definitions, we recall in Section 2 the PR constraints of cosine-modulated filter banks as derived by Nguyen et al. In Section 3, we derive from the PR constraints how to realize the filter bank using zero-delay and maximum-delay matrices. Section 4 shows that we can easily design filter banks with identical analysis and synthesis prototype filter by imposing constraints on the first matrix of the factorization. Section 5 shows how filter banks without DC leakage can be obtained by choosing the first matrix of the factorization appropriately. Section 6 compares the implementation cost of the new factorization with a direct implementation of the polyphase filters. In Section 7 we present design examples for low-delay prototype filters as well as for filter banks without DC leakage. Finally, Section 8 gives some conclusions.

1.1. Notation and definitions

Boldface letters denote matrices or vectors. Symbols $\boldsymbol{I}_M$ and $\boldsymbol{J}_M$ denote the $M \times M$ identity and counter-identity matrix, respectively.

The symbol $[\boldsymbol{F}]_{n,k}$ denotes the element at the nth row and kth column of the matrix $\boldsymbol{F}$. $\lceil x \rceil$ is the smallest integer greater than or equal to x, $\lfloor x \rfloor$ is the biggest integer smaller than or equal to x, and $\prod_{j=1}^{v} \boldsymbol{L}_j = \boldsymbol{L}_1 \cdot \ldots \cdot \boldsymbol{L}_v$, where the ordering is important. The notation $\boldsymbol{x}^t$ denotes the transpose of a vector $\boldsymbol{x}$. If the z-transform of a filter is given by $H(z)$, the corresponding impulse response in the time domain is denoted by $h(n)$.

The filter bank structure is shown in Fig. 1. The analysis filter bank consists of M parallel analysis filters of length N_h with impulse responses $h_k(n)$ and z-transforms $H_k(z)$, $k = 0, \ldots, M-1$, $n = 0, \ldots, N_h - 1$, and subsequent downsampling by M. The input signal is $x(n)$, and the subband signals are $y_k(m)$, $k = 0, \ldots, M-1$, where m is the time index at the reduced sampling rate. The synthesis filter bank consists of upsamplers by M followed by M synthesis filters of length N_f with impulse responses $f_k(n)$ and z-transforms $F_k(z)$, $k = 0, \ldots, M-1$, $n = 0, \ldots, N_f - 1$. The filter outputs are summed to form the reconstructed signal $\hat{x}(n)$. The filter bank provides perfect reconstruction if the output signal is a delayed version of the input signal, $\hat{x}(n) = x(n - D)$, where D is the system delay, i.e. the integer number of sampling periods that the output signal is delayed to the input signal, assuming that the subband signals are directly passed from the analysis to the synthesis bank. In the following, the system delay will be expressed as of $D = 2sM + d$ where s denotes the integer multiples of $2M$ and d the remainder of D and $2M$.

2. Cosine-modulated filter banks with perfect reconstruction

In this section, we recall the PR constraints from [1,8] for biorthogonal cosine-modulated filter banks. In [1] the generation of the M analysis and synthesis filters from the lowpass prototypes $H(z)$ and $F(z)$,

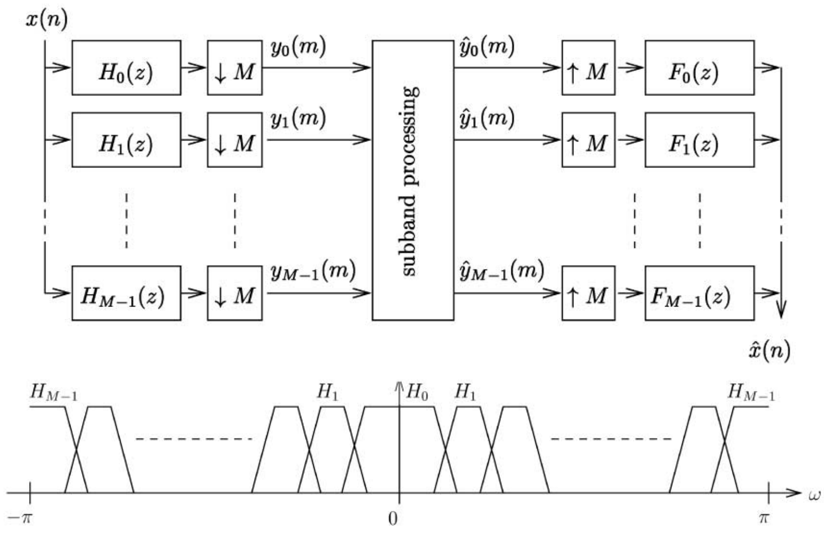

Fig. 1. Biorthogonal cosine-modulated filter bank.

respectively, has been chosen according to

$$h_k(n) = 2h(n)\cos\left(\frac{\pi}{M}(k+0.5)(n-D/2)+\theta_k\right),\quad n=0,1,\ldots,N_h-1, \tag{1}$$

$$f_k(n) = 2f(n)\cos\left(\frac{\pi}{M}(k+0.5)(n-D/2)-\theta_k\right),\quad n=0,1,\ldots,N_f-1, \tag{2}$$

where $\theta_k = (-1)^k\pi/4$. The filter lengths N_h and N_f, the number of subbands M, and the system delay D belong to the set of free parameters that has to be selected by the system designer according to the application.

We now express the analysis and synthesis prototype filters $H(z)$ and $F(z)$ by type-1 polyphase components $G_\ell(z)$ and $K_\ell(z)$, respectively, with $\ell = 0,\ldots,2M-1$:

$$H(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} G_\ell(z^{2M}),\quad g_\ell(m) = h(2mM+\ell), \tag{3}$$

$$F(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} K_\ell(z^{2M}),\quad k_\ell(m) = f(2mM+\ell). \tag{4}$$

Using these polyphase components, it has been demonstrated in [1] that the constraints mentioned below, which depend on the delay parameters d and s, have to be satisfied for the cosine-modulated filter bank to provide PR. See [1] for a more detailed derivation.

PR constraints for $0 \leqslant d < M$: The relation between the polyphase components of the analysis and synthesis prototype is

$$K_\ell(z) = \lambda_\ell z^{-a_\ell} G_\ell(z),\quad K_{\ell+M}(z) = \lambda_\ell z^{-a_\ell} G_{\ell+M}(z),\quad 0 \leqslant \ell < M,\ \ell \neq \frac{M+d}{2}, \tag{5}$$

where λ_ℓ denotes a non-zero, real valued scaling factor and a_ℓ an integer value describing a possible delay between the corresponding polyphase components of the analysis and synthesis prototype filters. They jointly have to satisfy

(1) $$G_\ell(z)K_{d-\ell}(z) + z^{-1}G_{M+\ell}(z)K_{M+d-\ell}(z) = \frac{z^{-s}}{2M},\quad 0 \leqslant \ell \leqslant d, \tag{6}$$

(2) $$G_\ell(z)K_{2M+d-\ell}(z) + G_{M+\ell}(z)K_{M+d-\ell}(z) = \frac{z^{-(s-1)}}{2M},\quad d < \ell < 2M,\ \ell \neq \frac{M+d}{2}, \tag{7}$$

(3) $\ell = (M+d)/2$:

$$G_{\frac{M+d}{2}}(z)K_{\frac{3M+d}{2}}(z) = \frac{z^{-(s-1)}}{4M},\quad K_{\frac{M+d}{2}}(z),\ G_{\frac{3M+d}{2}}(z)\ \text{arbitrary}\quad \text{for } s \text{ odd}, \tag{8}$$

$$G_{\frac{3M+d}{2}}(z)K_{\frac{M+d}{2}}(z) = \frac{-z^{-(s-1)}}{4M},\quad K_{\frac{3M+d}{2}}(z),\ G_{\frac{M+d}{2}}(z)\ \text{arbitrary}\quad \text{for } s \text{ even}. \tag{9}$$

PR constraints for $M \leqslant d < 2M$: The PR constraints for $M \leqslant d < 2M$ can be expressed in a similar way:

$$K_\ell(z) = \lambda_\ell z^{-a_\ell} G_\ell(z),\quad K_{\ell+M}(z) = \lambda_\ell z^{-a_\ell} G_{\ell+M}(z),\quad 0 \leqslant \ell < M,\ \ell \neq \frac{d-M}{2} \tag{10}$$

and

(1) $$G_\ell(z)K_{d-\ell}(z) + G_{M+\ell}(z)K_{d-\ell-M}(z) = \frac{z^{-s}}{2M}, \quad 0 \leqslant \ell \leqslant d-M,\ \ell \neq \frac{d-M}{2}, \tag{11}$$

(2) $$G_\ell(z)K_{d-\ell}(z) + z^{-1}G_{\ell+M}(z)K_{M+d-\ell}(z) = \frac{z^{-s}}{2M}, \quad d-M < \ell < 2M, \tag{12}$$

(3) $\ell = (d-M)/2$:

$$G_{\frac{M+d}{2}}(z)K_{\frac{d-M}{2}}(z) = \frac{-z^{-s}}{4M}, \quad K_{\frac{M+d}{2}}(z),\ G_{\frac{d-M}{2}}(z) \text{ arbitrary} \quad \text{for } s \text{ odd}, \tag{13}$$

$$G_{\frac{d-M}{2}}(z)K_{\frac{M+d}{2}}(z) = \frac{z^{-s}}{4M}, \quad K_{\frac{d-M}{2}}(z),\ G_{\frac{M+d}{2}}(z) \text{ arbitrary} \quad \text{for } s \text{ even}. \tag{14}$$

Some remarks on the PR constraints: From (5) and (10), it can be seen that the polyphase components of the analysis and synthesis prototype filters are strictly connected. I.e., they have to be equal up to the scale factors λ_ℓ and the delays a_ℓ. The value d of the overall system delay determines which polyphase filters are connected in the PR constraints (6)–(9) and (11)–(14), respectively, while s determines the delay on the right-hand side of the upper equations (remember that the overall system delay was given by $D = 2sM + d$). Note that for $d = 2M - 1$ and M being even, all PR constraints are given by (10) and (11).

3. Filter bank realization using zero-delay and maximum-delay matrices

A straightforward implementation of the biorthogonal cosine-modulated filter bank can be derived from the polyphase matrices. In [1] it has been shown that the analysis and synthesis polyphase matrices $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$, respectively, can be written as

$$\boldsymbol{E}(z) = \boldsymbol{C}_1 \begin{bmatrix} \boldsymbol{g}_0(-z^2) \\ z^{-1}\boldsymbol{g}_1(-z^2) \end{bmatrix}, \qquad \boldsymbol{R}(z) = [z^{-1}\boldsymbol{k}_1(-z^2) \quad \boldsymbol{k}_0(-z^2)]\boldsymbol{C}_2^{\mathrm{t}} \tag{15}$$

with

$$[\boldsymbol{C}_1]_{k,\ell} = 2\cos\left((k+0.5)\frac{\pi}{M}\left(\ell - \frac{D}{2}\right) + \theta_k\right), \quad 0 \leqslant k < M, \tag{16}$$

$$[\boldsymbol{C}_2]_{k,\ell} = 2\cos\left((k+0.5)\frac{\pi}{M}\left(2M - 1 - \ell - \frac{D}{2}\right) + \theta_k\right), \quad 0 \leqslant \ell < 2M \tag{17}$$

and

$$\boldsymbol{g}_0(-z^2) = \mathrm{diag}[G_0(-z^2), \ldots, G_{M-1}(-z^2)], \quad \boldsymbol{g}_1(-z^2) = \mathrm{diag}[G_M(-z^2), \ldots, G_{2M-1}(-z^2)],$$

$$\boldsymbol{k}_0(-z^2) = \mathrm{diag}[K_{M-1}(-z^2), \ldots, K_0(-z^2)], \quad \boldsymbol{k}_1(-z^2) = \mathrm{diag}[K_{2M-1}(-z^2), \ldots, K_M(-z^2)].$$

The corresponding polyphase realization of the filter bank is shown in Fig. 2. The input signal is split into M polyphase components. These components are fed into the $2M$ upsampled and modulated polyphase filters $G_\ell(-z^2)$. Their outputs are then transformed by the $2M \times M$ transform matrix $\boldsymbol{C}_1$ and yield the vector of M subband signals. On the synthesis side, mainly the inverse steps are performed. A similar realization has been derived in [2,16] for the paraunitary case.

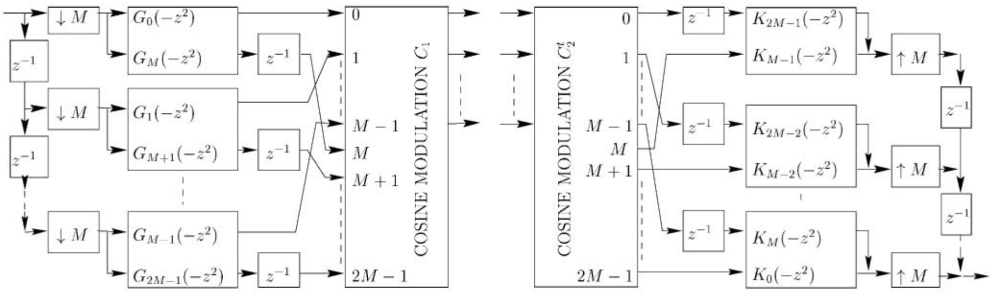

Fig. 2. Filter bank realization using polyphase filters $G_\ell(z)$ and $K_\ell(z)$, respectively, and cosine transform.

From the filter bank structure in Fig. 2, the following ideas arise for a more efficient realization:

1. The modulation matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are of size $M \times 2M$ and not of size $M \times M$ as in [4,12–15], which means that the complexity could be reduced by exploiting the properties of the modulation matrices.
2. The polyphase components $G_\ell(-z^2)$ and $G_{\ell+M}(-z^2)$ are fed with the same input signal, so that one could think of realizing always two polyphase filters jointly. Accordingly, on the synthesis side, the outputs of $K_\ell(-z^2)$ and $K_{\ell+M}(-z^2)$ are added, and both synthesis polyphase filters may also be realized jointly.

In the following, we show how these ideas result in an efficient realization. For simplicity, we only consider the case where d is in the range $M \leqslant d < 2M$. We have to treat the different PR constraints (11)–(14) separately. The derivations for $0 \leqslant d < M$ can be performed in an analog way.

3.1. Case1, $0 \leqslant \ell \leqslant d - M$, $\ell \neq \dfrac{d-M}{2}$

In the following, we regard the range $0 \leqslant \ell < (d-M)/2$. The results for $(d-M)/2 < \ell \leqslant d - M$ can be obtained from the ones derived here by substituting ℓ by $d - M - \ell$. By having a closer look at the modulation matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ from (16)–(17) one can verify that

$$[\boldsymbol{C}_1]_{k,d-M-\ell} = (-1)^s [\boldsymbol{C}_1]_{k,\ell}, \quad [\boldsymbol{C}_2]_{k,M-1-\ell} = (-1)^s [\boldsymbol{C}_2]_{k,2M-1-d+\ell}, \tag{18}$$

$$[\boldsymbol{C}_1]_{k,\ell+M} = (-1)^{s-1} [\boldsymbol{C}_1]_{k,d-\ell}, \quad [\boldsymbol{C}_2]_{k,3M-1-d+\ell} = (-1)^{s-1} [\boldsymbol{C}_2]_{k,2M-1-\ell}. \tag{19}$$

The polyphase matrices in (15) are now divided into subsystems $\boldsymbol{E}_\ell(z)$ and $\boldsymbol{R}_\ell(z)$ that contain the columns of the modulation matrices which are connected by the upper equations. For the analysis side we obtain

$$\boldsymbol{E}_\ell(z) = \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell-M} & c^1_{0,\ell+M} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell-M} & c^1_{1,\ell+M} & c^1_{1,d-\ell} \\ \vdots & & & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell-M} & c^1_{M-1,\ell+M} & c^1_{M-1,d-\ell} \end{bmatrix} \begin{bmatrix} G_\ell(-z^2) & 0 \\ 0 & G_{d-\ell-M}(-z^2) \\ z^{-1}G_{\ell+M}(-z^2) & 0 \\ 0 & z^{-1}G_{d-\ell}(-z^2) \end{bmatrix}$$

$$= \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell} \\ \vdots & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell} \end{bmatrix} \underbrace{\begin{bmatrix} G_\ell(-z^2) & (-1)^s G_{d-\ell-M}(-z^2) \\ (-1)^{s-1} z^{-1} G_{\ell+M}(-z^2) & z^{-1} G_{d-\ell}(-z^2) \end{bmatrix}}_{:= \boldsymbol{G}_\ell(z)} \tag{20}$$

with $c^1_{k,\ell} = [\boldsymbol{C}_1]_{k,\ell}$. Similarly, the subsystem $\boldsymbol{R}_\ell(z)$ of the synthesis polyphase matrix writes

$$\boldsymbol{R}_\ell(z) = \begin{bmatrix} z^{-1}K_{d-\ell}(-z^2) & 0 & K_{d-\ell-M}(-z^2) & 0 \\ 0 & z^{-1}K_{\ell+M}(-z^2) & 0 & K_\ell(-z^2) \end{bmatrix} \times \begin{bmatrix} c^2_{0,2M-1-d+\ell} & \cdots & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,M-1-\ell} & & c^2_{M-1,M-1-\ell} \\ c^2_{0,3M-1-d+\ell} & & c^2_{M-1,3M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \cdots & c^2_{M-1,2M-1-\ell} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} z^{-1}K_{d-\ell}(-z^2) & (-1)^{s-1}K_{d-\ell-M}(-z^2) \\ (-1)^s z^{-1}K_{\ell+M}(-z^2) & K_\ell(-z^2) \end{bmatrix}}_{:=\boldsymbol{K}_\ell(z)} \begin{bmatrix} c^2_{0,2M-1-d+\ell} & & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \cdots & c^2_{M-1,2M-1-\ell} \end{bmatrix} \tag{21}$$

with $c^2_{k,\ell} = [\boldsymbol{C}_2]_{k,\ell}$. The analysis and synthesis filter bank now can be realized as shown in Fig. 3. The modulation cost of this realization is reduced, because we have suppressed half the columns of the analysis and synthesis modulation submatrices. Note, however, that some further arrangements of the rows and columns of the new modulation matrices are needed before obtaining a form that can be realized by fast DCT. Also the polyphase filtering part has become more efficient, because we only have to compute the sum of two outputs of the analysis polyphase filters, instead of computing the output of all four polyphase filters explicitly.

When calculating $\boldsymbol{K}_\ell(z)\boldsymbol{G}_\ell(z)$ with $\boldsymbol{K}_\ell(z)$ and $\boldsymbol{G}_\ell(z)$ from (21) and (20), respectively, substituting $-z^2$ by z into the result and comparing the four entries of the matrix with the constraints for perfect reconstruction (10) and (11), it can be verified that the following relationship has to hold true for PR:

$$\boldsymbol{K}_\ell(z)\boldsymbol{G}_\ell(z) = \frac{(-1)^s z^{-2s-1}}{2M}\boldsymbol{I}_2, \quad 0 \leqslant \ell \leqslant d-M,\ \ell \neq \frac{d-M}{2}. \tag{22}$$

We may state the result as a theorem.

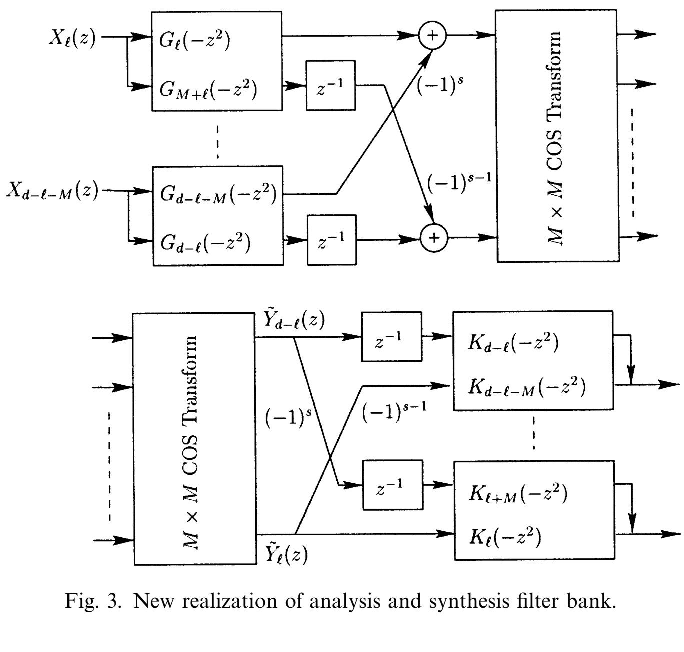

Fig. 3. New realization of analysis and synthesis filter bank.

Theorem. *Given a set of polyphase filters satisfying the PR constraint expressed by* (22) *the following realization for* $\boldsymbol{G}_\ell(z)$ *defined in* (20) *and* $\boldsymbol{K}_\ell(z)$ *according to* (21) *always exists*:

$$\boldsymbol{G}_\ell(z) = \prod_{j=1}^{j_0} \boldsymbol{D}_{\ell,j}(z) \prod_{i=1}^{i_0} \boldsymbol{B}_{\ell,i}(z) \cdot \boldsymbol{G}_{\ell,\mathrm{ini}}(z), \quad \boldsymbol{K}_\ell(z) = \boldsymbol{K}_{\ell,\mathrm{ini}}(z) \prod_{i=i_0}^{1} \boldsymbol{B}_{\ell,i}^{-1}(z) \prod_{j=j_0}^{1} (z^{-\delta_{\ell,j}-1}\boldsymbol{D}_{\ell,j}^{-1}(z)), \tag{23}$$

where j_0 can be computed from $s = 0.5\sum_{j=1}^{j_0}(\delta_{\ell,j}+1)$ for a fixed value of s in (22). The matrices in (23) are called *zero-delay matrices*, *maximum-delay matrices*, and *initialization matrices*.

- Zero-delay matrices:

$$\boldsymbol{B}_{\ell,i}(z) = \begin{bmatrix} 0 & 1 \\ 1 & b_{\ell,i}z^{-\beta_{\ell,i}} \end{bmatrix}, \quad \boldsymbol{B}_{\ell,i}^{-1}(z) = \begin{bmatrix} -b_{\ell,i}z^{-\beta_{\ell,i}} & 1 \\ 1 & 0 \end{bmatrix}. \tag{24}$$

- Maximum-delay matrices:

$$\boldsymbol{D}_{\ell,j}(z) = \begin{bmatrix} d_{\ell,j} & z^{-1} \\ z^{-\delta_{\ell,j}} & 0 \end{bmatrix}, \quad z^{-\delta_{\ell,j}-1}\boldsymbol{D}_{\ell,j}^{-1}(z) = \begin{bmatrix} 0 & z^{-1} \\ z^{-\delta_{\ell,j}} & -d_{\ell,j} \end{bmatrix}. \tag{25}$$

- Initialization matrices:

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = \begin{bmatrix} g_{\ell,0} & g_{\ell,1} \\ z^{-1}g_{\ell,2} & z^{-1}g_{\ell,3} \end{bmatrix}, \quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z) = \frac{1}{2M} \cdot \frac{(-1)^s}{g_{\ell,0}g_{\ell,3} - g_{\ell,1}g_{\ell,2}} \begin{bmatrix} g_{\ell,3}z^{-1} & -g_{\ell,1} \\ -g_{\ell,2}z^{-1} & g_{\ell,0} \end{bmatrix}. \tag{26}$$

  The variables $b_{\ell,i}$, $d_{\ell,j}$, $g_{\ell,0}$, $g_{\ell,1}$, $g_{\ell,2}$, $g_{\ell,3}$ are real valued coefficients and $\beta_{\ell,i}$, $\delta_{\ell,j}$ non-negative integer values.

The names of the matrices as well as their forms are adopted from [12–14]. However, we have newly introduced the variables $\beta_{\ell,i}$ and $\delta_{\ell,j}$ that allow the factorization of non-contiguous prototype impulse responses which was not the case in [12–14] where they were forced to be $\beta_{\ell,i} = \delta_{\ell,j} = 1$. The zero-delay matrices increase the filter length but not the delay. Thus, when creating new polyphase matrices $\boldsymbol{G}_\ell^{\mathrm{new}}(z)$ and $\boldsymbol{K}_\ell^{\mathrm{new}}(z)$ from given ones $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ according to

$$\boldsymbol{G}_\ell^{\mathrm{new}}(z) = \boldsymbol{B}_{\ell,i}(z) \cdot \boldsymbol{G}_\ell(z), \quad \boldsymbol{K}_\ell^{\mathrm{new}}(z) = \boldsymbol{K}_\ell(z) \cdot \boldsymbol{B}_{\ell,i}^{-1}(z), \tag{27}$$

where $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ satisfy (22), the polyphase components in the new polyphase matrices are longer than the original ones, but the delay s on right-hand side of (22) does not change: $\boldsymbol{K}_\ell^{\mathrm{new}}(z) \cdot \boldsymbol{G}_\ell^{\mathrm{new}}(z) = \boldsymbol{K}_\ell(z) \cdot \boldsymbol{G}_\ell(z)$. Introducing maximum delay matrices according to

$$\boldsymbol{G}_\ell^{\mathrm{new}}(z) = \boldsymbol{D}_{\ell,j}(z) \cdot \boldsymbol{G}_\ell(z), \quad \boldsymbol{K}_\ell^{\mathrm{new}}(z) = \boldsymbol{K}_\ell(z) \cdot z^{-\delta_{\ell,j}-1}\boldsymbol{D}_{\ell,j}^{-1}(z), \tag{28}$$

not only increases the length of the polyphase filters but also the delay on the right-hand side of (22) resulting in

$$\boldsymbol{K}_\ell^{\mathrm{new}}(z)\boldsymbol{G}_\ell^{\mathrm{new}}(z) = z^{-\delta_{\ell,j}-1}\boldsymbol{K}_\ell(z)\boldsymbol{G}_\ell(z) = \frac{(-1)^s z^{-2s-1-\delta_\ell-1}}{2M}\boldsymbol{I}_2. \tag{29}$$

3.1.1. Factorization of $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ into zero-delay, maximum-delay, and initialization matrices

- *Starting point*: We start with polyphase matrices $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ satisfying (22). Thus, the polyphase components in the matrices as described by (20) and (21), respectively, belong to PR analysis and synthesis prototype filters $H(z)$ and $F(z)$, respectively, designed for a cosine-modulated filter bank with overall system delay $D = 2sM + d$, where s and d have been fixed in the design process with $M \leqslant d < 2M$.

For reasons of conciseness, we just consider the case where $a_\ell = 0$ in (10) since all other choices for a_ℓ simply lead to a zero padding. The analysis and synthesis polyphase components $G_\ell(z)$ and $K_\ell(z)$, $\ell = 0,\ldots,2M-1$ then are identical up to a scalar factor.

For the following derivation, we define two counting variables i and j which are initialized as $i = j = 1$ and temporary matrices $\boldsymbol{V}_{G,\ell}(z)$ and $\boldsymbol{V}_{K,\ell}(z)$ which at the end of the derivation will contain the factorizations of $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$, respectively. They are initialized as $\boldsymbol{V}_{G,\ell}(z) = \boldsymbol{V}_{K,\ell}(z) = \boldsymbol{I}_2$. Furthermore, we call $s_0 = s$ the starting value of the delay parameter s.

- *Step 1*: *Factorization of $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ into maximum-delay matrices*

  If the delay parameter s_0 satisfies $s_0 > 0$, $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ can be factorized as

$$\boldsymbol{G}_\ell(z) = \boldsymbol{D}_j(z)\boldsymbol{G}_{\mathrm{sh}}(z), \quad \boldsymbol{K}_\ell(z) = \boldsymbol{K}_{\mathrm{sh}}(z)\boldsymbol{D}_j^{-1}(z)\cdot z^{-\delta_j-1}, \tag{30}$$

where $\boldsymbol{G}_\ell(z)$ is as in (20), $\boldsymbol{K}_\ell(z)$ is as in (21) and $\boldsymbol{D}_j(z)$ and $\boldsymbol{D}_j^{-1}(z)\cdot z^{-\delta_j-1}$ are as in (25). Note that for reasons of conciseness, we have dropped the subscript ℓ for the maximum delay matrices. $\boldsymbol{G}_{\mathrm{sh}}(z)$ and $\boldsymbol{K}_{\mathrm{sh}}(z)$ correspond to polyphase submatrices of shorter polyphase filters and are of temporary use during the factorization algorithm. They can be calculated as

$$\boldsymbol{G}_{\mathrm{sh}}(z) = \boldsymbol{D}_j^{-1}(z)\boldsymbol{G}_\ell(z), \quad \boldsymbol{K}_{\mathrm{sh}}(z) = \boldsymbol{K}_\ell(z)\boldsymbol{D}_j(z)\cdot z^{\delta_j+1} \tag{31}$$

and write

$$\boldsymbol{G}_{\mathrm{sh}}(z) = \begin{bmatrix} z^{\delta_j}[\boldsymbol{G}_\ell(z)]_{1,0} & z^{\delta_j}[\boldsymbol{G}_\ell(z)]_{1,1} \\ z[\boldsymbol{G}_\ell(z)]_{0,0} - d_j z^{\delta_j+1}[\boldsymbol{G}_\ell(z)]_{1,0} & z[\boldsymbol{G}_\ell(z)]_{0,1} - d_j z^{\delta_j+1}[\boldsymbol{G}_\ell(z)]_{1,1} \end{bmatrix}, \tag{32}$$

$$\boldsymbol{K}_{\mathrm{sh}}(z) = \begin{bmatrix} d_j z^{\delta_j+1}[\boldsymbol{K}_\ell(z)]_{0,0} + z[\boldsymbol{K}_\ell(z)]_{0,1} & z^{\delta_j}[\boldsymbol{K}_\ell(z)]_{0,0} \\ d_j z^{\delta_j+1}[\boldsymbol{K}_\ell(z)]_{1,0} + z[\boldsymbol{K}_\ell(z)]_{1,1} & z^{\delta_j}[\boldsymbol{K}_\ell(z)]_{1,0} \end{bmatrix}. \tag{33}$$

Let us denote by $g_{\nu,\mu}(n)$ and $k_{\nu,\mu}(n)$ the inverse z-transforms of $[\boldsymbol{G}_\ell(z)]_{\nu,\mu}$ and $[\boldsymbol{K}_\ell(z)]_{\nu,\mu}$, respectively, with $\nu,\mu = 0,1$. The matrices $\boldsymbol{G}_{\mathrm{sh}}(z)$ and $\boldsymbol{K}_{\mathrm{sh}}(z)$ have to be causal, which is satisfied if δ_j and d_j are chosen in the following way:

$$g_{1,0}(n) = g_{1,1}(n) = k_{0,0}(n) = k_{1,0}(n) = 0 \quad \forall n = 0,1,\ldots,\delta_j - 1, \tag{34}$$

$$g_{1,0}(\delta_j) \neq 0, \quad g_{1,1}(\delta_j) \neq 0, \quad k_{0,0}(\delta_j) \neq 0, \quad k_{1,0}(\delta_j) \neq 0, \tag{35}$$

$$d_j = \frac{g_{0,0}(0)}{g_{1,0}(\delta_j)} = \frac{g_{0,1}(0)}{g_{1,1}(\delta_j)} = -\frac{k_{0,1}(0)}{k_{0,0}(\delta_j)} = -\frac{k_{1,1}(0)}{k_{1,0}(\delta_j)}, \quad \delta_j \leqslant 2s_0 - 1. \tag{36}$$

In the appendix we show that all parts of the upper equalities hold true if the prototypes satisfy the PR constraints given in (10) and (11), which connects the PR constraints on the polyphase filters as given in [1] with the coefficients of the maximum-delay matrices introduced in [14]. The shorter polyphase matrices $\boldsymbol{G}_\ell^{\mathrm{sh}}(z)$ and $\boldsymbol{K}_\ell^{\mathrm{sh}}(z)$ satisfy

$$\boldsymbol{K}_\ell^{\mathrm{sh}}(z)\boldsymbol{G}_\ell^{\mathrm{sh}}(z) = (-1)^s \frac{z^{-1-2s+\delta_j+1}}{2M} \tag{37}$$

which means that their entries can again be interpreted as polyphase filters satisfying the PR constraints but now for a shorter delay $s_0 - (\delta_j + 1)/2$ which always is an integer value since δ_j can only take odd values. We now set

$$\boldsymbol{V}_{G,\ell}(z) := \boldsymbol{D}_j(z)\boldsymbol{V}_{G,\ell}(z), \quad \boldsymbol{V}_{K,\ell}(z) := \boldsymbol{V}_{K,\ell}(z)z^{-\delta_j-1}\boldsymbol{D}_j^{-1}(z), \tag{38}$$

$$G_\ell(z) := G_{\text{sh}}(z), \quad K_\ell(z) := K_{\text{sh}}(z), \tag{39}$$

$$s_0 := s_0 - (\delta_j + 1)/2, \quad j := j + 1 \tag{40}$$

and restart from the top of *Step 1* as long as $s_0 > 0$.

- *Step 2*: Once we arrive at $s_0 = 0$ we cannot reduce the delay any further. We now set $j_0 = j - 1$. The matrices $V_{G,\ell}(z)$ and $V_{K,\ell}(z)$ by now write

$$V_{G,\ell}(z) = \prod_{j=1}^{j_0} D_j(z), \quad V_{K,\ell}(z) = \prod_{j=j_0}^{1} (D_j^{-1}(z) z^{-\delta_j - 1}) \tag{41}$$

and the product $K_\ell(z) \cdot G_\ell(z)$ is given by

$$K_\ell(z) \cdot G_\ell(z) = \frac{(-1)^s z^{-1}}{2M} I_2. \tag{42}$$

- *Step 3*: *Factorization of $G_\ell(z)$ and $K_\ell(z)$ into zero-delay matrices*
  As long as $G_\ell(z)$ and $K_\ell(z)$ have more than one non-zero entry (which for $[G_\ell(z)]_{\nu,\mu}$ are $g_{0,0}(0)$, $g_{0,1}(0)$, $g_{1,0}(1)$, $g_{1,1}(1)$ and for $[K_\ell(z)]_{\nu,\mu}$ $k_{0,0}(1)$ $k_{0,1}(0)$, $k_{1,0}(1)$ $k_{1,1}(0)$), we can still factor $G_\ell(z)$ and $K_\ell(z)$ into zero-delay matrices and thus reduce the filter length (otherwise we directly proceed to Step 4).
  We perform in the same way as before with the only difference that we now split zero-delay matrices $B_i(z)$ from $G_\ell(z)$ and $K_\ell(z)$. The index i is the counting variable defined at the starting point. Again, we omit the subscript ℓ for the zero-delay matrices. $G_\ell(z)$ and $K_\ell(z)$ are factorized as

$$G_\ell(z) = B_i(z) G_{\text{sh}}(z), \quad K_\ell(z) = K_{\text{sh}}(z) B_i^{-1}(z), \tag{43}$$

$G_{\text{sh}}(z)$ and $K_{\text{sh}}(z)$ can be written as

$$G_{\text{sh}}(z) = \begin{bmatrix} [G_\ell(z)]_{1,0} - b_i z^{-\beta_i}[G_\ell(z)]_{0,0} & [G_\ell(z)]_{1,1} - b_i z^{-\beta_i}[G_\ell(z)]_{0,1} \\ [G_\ell(z)]_{0,0} & [G_\ell(z)]_{0,1} \end{bmatrix}, \tag{44}$$

$$K_{\text{sh}}(z) = \begin{bmatrix} [K_\ell(z)]_{0,1} & [K_\ell(z)]_{0,0} + b_i z^{-\beta_i}[K_\ell(z)]_{0,1} \\ [K_\ell(z)]_{1,1} & [K_\ell(z)]_{1,0} + b_i z^{-\beta_i}[K_\ell(z)]_{1,1} \end{bmatrix}. \tag{45}$$

Since our aim in this step is to shorten the length of the entries in $G_\ell(z)$ and $K_\ell(z)$, the matrices $G_{\text{sh}}(z)$ and $K_{\text{sh}}(z)$ must not contain longer entries than $G_\ell(z)$ and $K_\ell(z)$. Denoting by $N_{g,\nu,\mu}$ the length of $[G_\ell(z)]_{\nu,\mu}$ and by $N_{k,\nu,\mu}$ the length of $[K_\ell(z)]_{\nu,\mu}$, we obtain for the parameters β_i and b_i the following solution:

$$\beta_i = N_{g,1,0} - N_{g,0,0} = N_{g,1,1} - N_{g,0,1} = N_{k,0,0} - N_{k,0,1} = N_{k,0,0} - N_{k,0,1} = N_{k,1,0} - N_{k,1,1}, \tag{46}$$

$$b_i = \frac{g_{1,0}(N_{g,1,0} - 1)}{g_{0,0}(N_{g,0,0} - 1)} = \frac{g_{1,1}(N_{g,1,1} - 1)}{g_{0,1}(N_{g,0,1} - 1)} = -\frac{k_{0,0}(N_{k,0,0} - 1)}{k_{0,1}(N_{k,0,1} - 1)} = -\frac{k_{1,0}(N_{k,1,0} - 1)}{k_{1,1}(N_{k,1,1} - 1)}. \tag{47}$$

We show in the appendix that all parts of the upper two equations are fulfilled for PR prototypes satisfying the constraints from [1] given in (10) and (11). Thus, (47) establishes the relationship between the direct formulation of the PR constraints on the polyphase filters from [1] and the coefficients of the zero-delay matrices from [14]. Note that we only get a causal solution for $\beta_i \geqslant 0$. If it turns out that $\beta_i < 0$ in (46) we choose $b_i = 0$ and thus $B_i(z) = J_2$, which means that we just flip the position of the entries in $G_\ell(z)$ and $K_\ell(z)$. Then, in the next iteration we obtain a positive β_i. Finally, we can set

$$V_{G,\ell}(z) = B_i(z) V_{G,\ell}(z), \quad V_{K,\ell}(z) = V_{K,\ell}(z) B_i^{-1}(z), \tag{48}$$

$$G_\ell(z) = G_{\text{sh}}(z), \quad K_\ell(z) = K_{\text{sh}}(z), \quad i = i + 1. \tag{49}$$

While the new matrices $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ still have more than one non-zero entry at each position, we can still continue factorizing the matrices into zero-delay matrices and restart from the top of Step 3.

- *Step 4*: The polyphase matrices $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ now have one of the following forms:

$$\boldsymbol{G}_\ell(z) = \begin{bmatrix} g_{0,0}(1)z^{-1} & g_{0,1}(1)z^{-1} \\ g_{1,0}(0) & g_{1,1}(0) \end{bmatrix}, \quad \boldsymbol{K}_\ell(z) = \begin{bmatrix} k_{0,0}(1)z^{-1} & k_{0,0}(0) \\ k_{1,0}(1)z^{-1} & k_{1,1}(0) \end{bmatrix} \tag{50}$$

or

$$\boldsymbol{G}_\ell(z) = \begin{bmatrix} g_{0,0}(0) & g_{0,1}(0) \\ g_{1,0}(1)z^{-1} & g_{1,1}(1)z^{-1} \end{bmatrix}, \quad \boldsymbol{K}_\ell(z) = \begin{bmatrix} k_{0,0}(0) & k_{0,1}(1)z^{-1} \\ k_{1,0}(0) & k_{1,1}(1)z^{-1} \end{bmatrix}. \tag{51}$$

The matrices in (51) already have the form of the initialization matrices given in (26). If Step 3 ends with the matrices i (50) we introduce, in an additional step, a counter identity matrix to bring them in the desired form (51), i.e. we perform

$$\boldsymbol{V}_{G,\ell}(z) := \boldsymbol{J}_2 \boldsymbol{V}_{G,\ell}(z), \quad \boldsymbol{V}_{K,\ell}(z) := \boldsymbol{V}_{K,\ell}(z)\boldsymbol{J}_2, \tag{52}$$

$$\boldsymbol{G}_\ell(z) := \boldsymbol{G}_\ell(z)\boldsymbol{J}_2, \quad \boldsymbol{K}_\ell(z) := \boldsymbol{J}_2\boldsymbol{K}_\ell(z), \quad i := i+1. \tag{53}$$

Setting $i_0 = (i-1)$, the matrices $\boldsymbol{V}_{G,\ell}(z)$ and $\boldsymbol{V}_{K,\ell}(z)$ then can be written as

$$\boldsymbol{V}_{G,\ell}(z) = \prod_{i=1}^{2i_0} \boldsymbol{B}_i(z) \prod_{j=1}^{j_0} \boldsymbol{D}_j(z), \quad \boldsymbol{V}_{K,\ell}(z) = \prod_{j=j_0}^{1} \boldsymbol{D}_j^{-1}(z) z^{-2\delta_j} \prod_{i=2i_0}^{1} \boldsymbol{B}_i^{-1}(z). \tag{54}$$

Thus, they contain the desired factorization. In a terminating step we set the initialization matrices to

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = \boldsymbol{G}_\ell(z), \quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z) = \boldsymbol{K}_\ell(z). \tag{55}$$

Note. We have derived the factorization for a certain sequence of zero-delay and maximum-delay matrices. However, a factorization is guaranteed for any arbitrary order of zero-delay and maximum-delay matrices.

3.2. Case 2, $d - M < \ell < 2M$

The derivation for this case is very similar to the one described above and will therefore only be sketched. Again, we just regard the range $d - M < \ell \leqslant d/2$, because the other half can easily be obtained when substituting ℓ by $d - M - \ell$.

The modulation matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ here satisfy

$$[\boldsymbol{C}_1]_{k,M+d-\ell} = (-1)^{s-1}[\boldsymbol{C}_1]_{k,\ell}, \quad [\boldsymbol{C}_2]_{k,M-1-\ell} = (-1)^{s}[\boldsymbol{C}_2]_{k,2M-1-d+\ell}, \tag{56}$$

$$[\boldsymbol{C}_1]_{k,\ell+M} = (-1)^{s-1}[\boldsymbol{C}_1]_{k,d-\ell}, \quad [\boldsymbol{C}_2]_{k,M-1-d+\ell} = (-1)^{s}[\boldsymbol{C}_2]_{k,2M-1-\ell}. \tag{57}$$

We see that always two columns of these matrices are identical up to the sign, so that the submatrices of the analysis and synthesis polyphase matrices can be written as

$$\boldsymbol{E}_\ell(z) = \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell} \\ \vdots & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell} \end{bmatrix} \underbrace{\begin{bmatrix} G_\ell(-z^2) & (-1)^{s-1}z^{-1}G_{M+d-\ell}(-z^2) \\ (-1)^{s-1}z^{-1}G_{\ell+M}(-z^2) & G_{d-\ell}(-z^2) \end{bmatrix}}_{:=\boldsymbol{G}_\ell(z)} \tag{58}$$

and

$$\boldsymbol{R}_\ell(z) = \underbrace{\begin{bmatrix} K_{d-\ell}(-z^2) & (-1)^s z^{-1} K_{M+d-\ell}(-z^2) \\ (-1)^s z^{-1} K_{\ell+M}(-z^2) & K_\ell(-z^2) \end{bmatrix}}_{:=\boldsymbol{K}_\ell(z)} \begin{bmatrix} c^2_{0,2M-1-d+\ell} & & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \cdots & c^2_{M-1,2M-1-\ell} \end{bmatrix}. \tag{59}$$

The matrices $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ are slightly different from the ones in Case 1, because here the delays z^{-1} are placed on the anti-diagonal. Furthermore, the signs of the terms on the anti-diagonal are equal. Comparing the result for the product $\boldsymbol{K}_\ell(z)\boldsymbol{G}_\ell(z)$ with the PR constraints (10) and (12), we obtain the following relationship:

$$\boldsymbol{K}_\ell(z)\boldsymbol{G}_\ell(z) = \frac{(-z^{-2})^s}{2M}\boldsymbol{I}_2, \quad d - M < \ell < 2M. \tag{60}$$

The factorization into maximum-delay and zero-delay matrices is mostly the same as in Case 1 using the same argumentation for the choice of β_i, b_i, δ_j, and d_j. Due to the different structure of $\boldsymbol{G}_\ell(z)$ and $\boldsymbol{K}_\ell(z)$ also the initialization matrices $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$ and $\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$ differ from the ones in Case 1. In the most general form $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$ and its inverse can be written as

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = \begin{bmatrix} g_{\ell,0} & z^{-1}g_{\ell,1} \\ z^{-1}g_{\ell,2} & g_{\ell,3} \end{bmatrix}, \quad \boldsymbol{G}^{-1}_{\ell,\mathrm{ini}}(z) = \frac{1}{g_{\ell,0}g_{\ell,3} - z^{-2}g_{\ell,1}g_{\ell,2}} \begin{bmatrix} g_{\ell,3} & -g_{\ell,1}z^{-1} \\ -g_{\ell,2}z^{-1} & g_{\ell,0} \end{bmatrix}, \tag{61}$$

$\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$ is a delayed version of $\boldsymbol{G}^{-1}_{\ell,\mathrm{ini}}(z)$, scaled by $(-1)^s/2M$. Since we restrict ourselves to the case where all filters in $\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$ are causal and FIR, we see from (61) that at least one of the coefficients in $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$ has to be zero. Thus, we obtain the following solutions for $\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$.

- If $g_{\ell,0} = 0$ or $g_{\ell,3} = 0$:

$$\boldsymbol{K}_{\ell,\mathrm{ini}}(z) = \frac{(-1)^{s-1}}{2Mg_{\ell,1}g_{\ell,2}} \begin{bmatrix} g_{\ell,3} & -g_{\ell,1}z^{-1} \\ -g_{\ell,2}z^{-1} & g_{\ell,0} \end{bmatrix}, \quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z)\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = (-1)^s z^{-2}\boldsymbol{I}_2. \tag{62}$$

- If $g_{\ell,1} = 0$ or $g_{\ell,2} = 0$:

$$\boldsymbol{K}_{\ell,\mathrm{ini}}(z) = \frac{(-1)^{s}}{2Mg_{\ell,0}\,g_{\ell,3}} \begin{bmatrix} g_{\ell,3} & -g_{\ell,1}z^{-1} \\ -g_{\ell,2}z^{-1} & g_{0} \end{bmatrix}, \quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z)\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = (-1)^s\boldsymbol{I}_2. \tag{63}$$

For the upper solution we have to stop factorizing maximum-delay matrices when $s = 1$ and keep the last delay for the initialization matrices.

The case $\ell = d/2$ with d being even needs special consideration. In this case, the matrices $\boldsymbol{G}_{d/2}(z)$ and $\boldsymbol{K}_{d/2}(z)$ can be written as

$$\boldsymbol{G}_{d/2}(z) = \begin{bmatrix} G_{d/2}(-z^2) & (-1)^{s-1}z^{-1}G_{d/2+M}(-z^2) \\ (-1)^{s-1}z^{-1}G_{d/2+M}(-z^2) & G_{d/2}(-z^2) \end{bmatrix}, \tag{64}$$

$$\boldsymbol{K}_{d/2}(z) = \begin{bmatrix} K_{d/2}(-z^2) & (-1)^{s}z^{-1}K_{d/2+M}(-z^2) \\ (-1)^{s}z^{-1}K_{d/2+M}(-z^2) & K_{d/2}(-z^2) \end{bmatrix}. \tag{65}$$

Both matrices are Toeplitz and the only possible cascades that keep this structure using the matrices $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$ and $\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$ in (61)–(63) and the zero-delay and maximum-delay matrices in (24) and (25) are given by

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z)=\begin{bmatrix}0 & g_{\ell,1}z^{-1}\\ g_{\ell,1}z^{-1} & 0\end{bmatrix},\quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z)=\frac{(-1)^{s-1}}{2Mg_{\ell,1}^2}\begin{bmatrix}0 & -g_{\ell,1}z^{-1}\\ -g_{\ell,1}z^{-1} & 0\end{bmatrix},$$
$$\boldsymbol{D}_{\ell,1}(z)=\begin{bmatrix}0 & z^{-1}\\ z^{-2s+1} & 0\end{bmatrix} \tag{66}$$

and

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z)=g_{\ell,0}\boldsymbol{I}_2,\quad \boldsymbol{K}_{\ell,\mathrm{ini}}(z)=\frac{(-1)^s}{2Mg_{\ell,0}}\boldsymbol{I}_2,\quad \boldsymbol{D}_{\ell,1}(z)=\begin{bmatrix}0 & z^{-1}\\ z^{-2s+3} & 0\end{bmatrix}. \tag{67}$$

3.3. Case 3, $\ell=(d-M)/2$

For $\ell=(d-M)/2$, the PR constraints, as given in (14), do not have the same form as in the other cases discussed above. Here two polyphase components can be chosen arbitrarily. This is due to the fact that the output of these analysis polyphase components will be multiplied with zero in the transform and the synthesis polyphase component is fed with a subband signal which is identically zero. Thus, these two polyphase filters do not have any influence in the filter bank and can be omitted in the realization, resulting in the lowest implementation cost. The remaining two filters can only have one non-zero coefficient each.

3.4. Notation for the special case $D=2sM+2M-1$

In this special case, the PR constraints for all polyphase components can be expressed by (10) and (11). Using the modifications discussed in this section, the analysis and synthesis polyphase matrices $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ can be written as

$$\boldsymbol{E}(z)=\tilde{\boldsymbol{C}}_1\boldsymbol{G}(z),\quad \boldsymbol{R}(z)=\boldsymbol{K}(z)\tilde{\boldsymbol{C}}_2^{\mathrm{t}} \tag{68}$$

with

$$[\tilde{\boldsymbol{C}}_1]_{k,\ell}=[\boldsymbol{C}_1]_{k,\ell},\quad [\tilde{\boldsymbol{C}}_1]_{k,M-1-\ell}=[\boldsymbol{C}_1]_{k,2M-1-\ell},\quad 0\leqslant k<M,\ 0\leqslant \ell<M/2, \tag{69}$$

$$[\tilde{\boldsymbol{C}}_2]_{k,\ell}=[\boldsymbol{C}_2]_{k,\ell},\quad [\tilde{\boldsymbol{C}}_2]_{k,M-1-\ell}=[\boldsymbol{C}_2]_{k,2M-1-\ell},\quad 0\leqslant k<M,\ 0\leqslant \ell<M/2, \tag{70}$$

$$\begin{aligned}\boldsymbol{G}(z)={}&\mathrm{diag}([G_0(-z^2),\ldots,G_{M/2-1}(-z^2),z^{-1}G_{3M/2}(-z^2),\ldots,z^{-1}G_{2M-1}(-z^2)])\\&+(-1)^s\boldsymbol{J}_M\cdot\mathrm{diag}([-z^{-1}G_M(-z^2),\ldots,-z^{-1}G_{3M/2-1}(-z^2),G_{M/2}(-z^2),\ldots,G_{M-1}(-z^2)])\end{aligned} \tag{71}$$

$$\begin{aligned}\boldsymbol{K}(z)={}&\mathrm{diag}([z^{-1}K_{2M-1}(-z^2),\ldots,z^{-1}K_{3M/2}(-z^2),K_{M/2-1}(-z^2),\ldots,K_0(-z^2)])\\&+(-1)^s\boldsymbol{J}_M\cdot\mathrm{diag}([K_M(-z^2),\ldots,K_{3M/2-1}(-z^2),-K_{M/2}(-z^2),\ldots,-K_{M-1}(-z^2)]).\end{aligned} \tag{72}$$

We have $\tilde{\boldsymbol{C}}_1=(-1)^s\tilde{\boldsymbol{C}}_2$ and $\tilde{\boldsymbol{C}}_1^{-1}\sqrt{2M}=\tilde{\boldsymbol{C}}_1^{\mathrm{t}}$. However, similar results can be obtained for arbitrary delays when taking into consideration the results obtained from (12) and (14).

4. Design of identical analysis and synthesis filters

We started with a general approach where two different prototypes were considered: one for the analysis and one for the synthesis side. However, from (10) we have seen that both prototypes are highly related to each other in the case of a PR filter bank. In fact, the prototypes' polyphase components have to be equal up to a scalar factor and a delay. Thus, the design freedom for obtaining different analysis and synthesis filters which both have the desired frequency responses is relatively small. Therefore, one often only considers the case where the analysis and synthesis prototypes are identical:

$$K_\ell(z) = G_\ell(z), \quad \ell = 0, \ldots, 2M-1. \tag{73}$$

Then, for Case 1, (11) can be expressed by the analysis polyphase filters only:

$$z^{-1}G_\ell(-z^2)G_{d-\ell}(-z^2) + z^{-1}G_{d-\ell-M}(-z^2)G_{\ell+M}(-z^2) = \det(\boldsymbol{G}_\ell(z)) = \frac{(-z^{-2})^s z^{-1}}{2M}. \tag{74}$$

Realizing $\boldsymbol{G}_\ell(z)$ as in (23), we may write $\det(\boldsymbol{G}_\ell(z))$ as

$$\det(\boldsymbol{G}_\ell(z)) = \prod_{j=1}^{j_0} \det(\boldsymbol{D}_{\ell,j}(z)) \cdot \prod_{i=1}^{i_0} \det(\boldsymbol{B}_{\ell,i}(z)) \cdot \det(\boldsymbol{G}_{\ell,\mathrm{ini}}(z)). \tag{75}$$

Using the properties $\det(\boldsymbol{B}_{\ell,i}(z)) = 1$ and $\det(\boldsymbol{D}_{\ell,j}(z)) = z^{-\delta_{\ell,j}-1}$, we obtain

$$\det(\boldsymbol{G}_\ell(z)) = \prod_{j=1}^{j_0} (z^{-\delta_{\ell,j}} - 1)\det(\boldsymbol{G}_{\ell,\mathrm{ini}}(z)) \overset{!}{=} \frac{z^{-1}(-z^{-2})^s}{2M}. \tag{76}$$

From $\sum_j (\delta_j + 1) = 2s$ we finally get

$$\det(\boldsymbol{G}_{\ell,\mathrm{ini}}(z)) = z^{-1}(g_{\ell,0}g_{\ell,3} - g_{\ell,1}\,g_{\ell,2}) \overset{!}{=} (-1)^s \frac{z^{-1}}{2M}. \tag{77}$$

The relationship (77) is satisfied with

$$\boldsymbol{G}_{\ell,\mathrm{ini}}(z) = \frac{(-1)^s}{2M}\begin{bmatrix} 1 & 0 \\ \tilde{g}_{\ell,0}z^{-1} & z^{-1} \end{bmatrix}\begin{bmatrix} 1 & \tilde{g}_{\ell,1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \tilde{g}_{\ell,2} & 1 \end{bmatrix}. \tag{78}$$

In this case, we have the advantage that $\boldsymbol{K}_{\ell,\mathrm{ini}}(z)$ contains the same coefficients as $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$. It writes

$$\boldsymbol{K}_{\ell,\mathrm{ini}}(z) = \frac{1}{2M}\begin{bmatrix} 1 & 0 \\ -\tilde{g}_{\ell,2} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tilde{g}_{\ell,1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z^{-1} & 0 \\ -\tilde{g}_{\ell,0}z^{-1} & 1 \end{bmatrix}. \tag{79}$$

5. Biorthogonal cosine-modulated filter banks without DC leakage

When processing signals with a DC component (e.g. images), it is important to use filter banks without DC leakage, meaning that the DC component of the input signal only affects the lowpass subband signal. Otherwise, artifacts such as the checkerboard effect may occur when quantizing the subband signals. Fig. 4 demonstrates this phenomenon for a gray scale image containing only a DC component. The input signal is split into $M = 8$ subbands using two different sets of analysis filters (with and without DC leakage, see Section 7 for the prototype filters). Then, all subbands apart from the lowpass band, where we expect the signal to be located, are suppressed and the image is reconstructed, resulting once in an image with visible checkerboard artifacts and once in a perfectly reconstructed single color image.

A filter bank is free of DC leakage if all analysis filters apart from the lowpass filter have at least one zero at $z = 1$. For the biorthogonal cosine-modulated filter bank this means that $H_k(1) = 0$ for $k = 1, \ldots, M-1$, while the lowpass filter has to satisfy $H_0(1) = 1$.

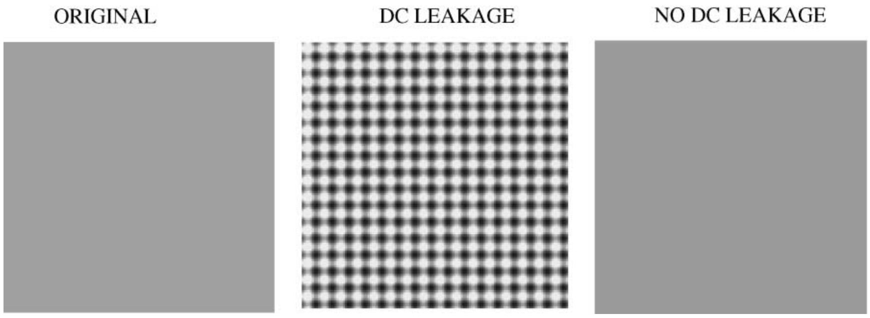

Fig. 4. Original image and reconstructed images using filter banks with and without DC leakage.

Let us consider the vector $\boldsymbol{h}(z) = [H_0(z), \ldots, H_{M-1}(z)]^{\mathrm{t}}$, which can be obtained from the analysis polyphase matrix as

$$\boldsymbol{h}(z) = \boldsymbol{E}(z^M)[1, z^{-1}, \ldots, z^{-(M-1)}]^{\mathrm{t}}. \tag{80}$$

We are interested in the DC behavior ($z = 1$), for which the upper equation can be written as

$$[1,0, \ldots ,0]^{\mathrm{t}} = \boldsymbol{E}(z^M)[1,1, \ldots ,1]^{\mathrm{t}}|_{z=1}. \tag{81}$$

For the reason of conciseness we only consider the case $D = 2sM + 2M - 1$. The analysis polyphase matrix has the form (68). Thus, we get the following linear system of equations:

$$\tilde{\boldsymbol{C}}_1^{-1}[1,0, \ldots ,0]^{\mathrm{t}} = \boldsymbol{G}(z)[1,1, \ldots ,1]^{\mathrm{t}}|_{z=1}. \tag{82}$$

Splitting $\boldsymbol{G}(z)$ into its submatrices $\boldsymbol{G}_\ell(z)$, $\ell = 0, \ldots, M/2 - 1$, and taking into consideration that $\boldsymbol{G}_\ell(z)$ can be realized by the cascade (23), Eq. (82) yields

$$\begin{bmatrix} [\tilde{\boldsymbol{C}}_1^{-1}]_{\ell,0} \\ [\tilde{\boldsymbol{C}}_1^{-1}]_{d-\ell,0} \end{bmatrix} = \prod_{j=1}^{j_0} \begin{bmatrix} d_{\ell,j} & 1 \\ 1 & 0 \end{bmatrix} \prod_{i=1}^{i_0} \begin{bmatrix} 0 & 1 \\ 1 & b_{\ell,i} \end{bmatrix} \cdot \begin{bmatrix} g_{\ell,0} & g_{\ell,1} \\ g_{\ell,2} & g_{\ell,3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{83}$$

and thus

$$\begin{bmatrix} g_{\ell,0} + g_{\ell,1} \\ g_{\ell,2} + g_{\ell,3} \end{bmatrix} = \prod_{i=i_0}^{1} \begin{bmatrix} -b_{\ell,i} & 1 \\ 1 & 0 \end{bmatrix} \prod_{j=j_0}^{1} \begin{bmatrix} 0 & 1 \\ 1 & -d_{\ell,i} \end{bmatrix} \cdot \begin{bmatrix} [\tilde{\boldsymbol{C}}_1^{-1}]_{\ell,0} \\ [\tilde{\boldsymbol{C}}_1^{-1}]_{2M-1-\ell,0} \end{bmatrix}. \tag{84}$$

We see that designing filter banks without DC leakage is possible by imposing constraint (84) on $\boldsymbol{G}_{\ell,\mathrm{ini}}(z)$. This only slightly reduces the parameter space for the filter optimization.

6. Implementation cost

In this section, we compare the implementation cost for the direct implementation of the polyphase components to the cost for an implementation by cascading zero-delay and maximum-delay matrices. We start with looking at the cost for the direct polyphase filter implementation as shown in Fig. 2, considering an overall system delay of the form $D = 2sM + 2M - 1$. In this case we need $2mM$ multiplications and $2(m-1)M$ additions for a prototype filter of length $N = 2mM$. When realizing the polyphase filters with zero-delay and maximum-delay matrices, we need $M/2$ matrices $\boldsymbol{G}_\ell(z)$ in parallel for $\ell = 0, \ldots, M/2 - 1$. We here assume that the prototype impulse response is contiguous, i.e. that all δ_j and β_i have been chosen equal to 1 which is the worst case in terms of implementation cost. The implementation cost for each $\boldsymbol{G}_\ell$ then is as follows: The first step of the iteration, i.e. the product of $\boldsymbol{G}_{\ell,\text{ini}}(z)$ from (26) with the input samples needs four multiplications and two additions. In order to obtain polyphase filters of length m, the cascade contains $j_0 + i_0 = 2(m-1)$ zero-delay and maximum-delay matrices that each can be realized with 1 multiplication and 1 addition. Thus, the implementation cost for the filtering part using zero-delay and maximum-delay matrices is

$$(4 + 2m - 2)M/2 = (m + 1)M \quad \text{multiplications,} \tag{85}$$

$$(2 + 2m - 2)M/2 = mM \quad \text{addition} \tag{86}$$

which is approximately half the implementation cost of the original polyphase filtering. Since the same matrix coefficients are found in the analysis and in the synthesis cascade, a coefficient quantization does not change the inherent PR property of the realization. Thus, we have the freedom to optimize the coefficients not only with regard to the frequency response, but also with regard to an efficient hardware (e.g. VLSI) implementation.

7. Prototype filter design using zero-delay and maximum-delay matrices

In this section we show, how the polyphase filter implementation using a cascade of zero-delay and maximum-delay matrices can be applied to the design of low-delay prototype filters as well as prototype filters without any DC leakage.

7.1. Analysis filters without DC leakage

In the first example we demonstrate how to apply the new filter implementation in order to achieve a cosine-modulated filter banks without DC leakage. For the image compression example in Fig. 4 we used a PR linear-phase prototype filter that was designed using the quadratic-constrained least-squares (QCSL) algorithm from [7] for the filter bank with DC leakage. The number of subbands is $M = 8$, the filter length is $N = 32$ and the same prototype filter is applied on the analysis and synthesis sides. Since the prototype is linear phase, the overall system delay writes $D = N - 1 = 31$. The frequency response of the cosine-modulated filter bank derived from this prototype is shown in the left-hand side of Fig. 5. To obtain a cosine-modulated filter bank without DC leakage, we take this prototype filter as a starting point. Since the delay is $D = 31 = 2M + 15$, we get $s = 1$ and $d = 15$ and have to consider the PR constraints (10) and (11), which are in this case valid for all $\ell = 0, \ldots, 2M - 1$. This implies that we have to use the factorization into zero-delay and maximum-delay matrices as given in (23), calculating its coefficients according to (36), (47), and (55) from the given polyphase filters. The factorization can be written as

$$\boldsymbol{G}_\ell(z) = \boldsymbol{D}_{\ell,1}(z) \cdot \boldsymbol{J}_2 \cdot \boldsymbol{B}_{\ell,2}(z) \cdot \boldsymbol{J}_2 \cdot \boldsymbol{G}_{\ell,\text{ini}}(z) \tag{87}$$

with coefficients given in Table 1. Then, changing the initialization matrices $\boldsymbol{G}_{\ell,\text{ini}}(z)$ from their general form in such a way that they satisfy (84) and keeping all the other matrices identical we obtain the analysis filters as shown in the right-hand side of Fig. 5 and the coefficients for the initialization matrices in Table 1. It can be easily seen that all analysis filters but the lowpass filter have a zero at frequency zero. Apart from this, the frequency responses are very similar to the starting point with DC leakage. Thus, the frequency selectivity does not worsen when imposing the zeros at frequency zero.

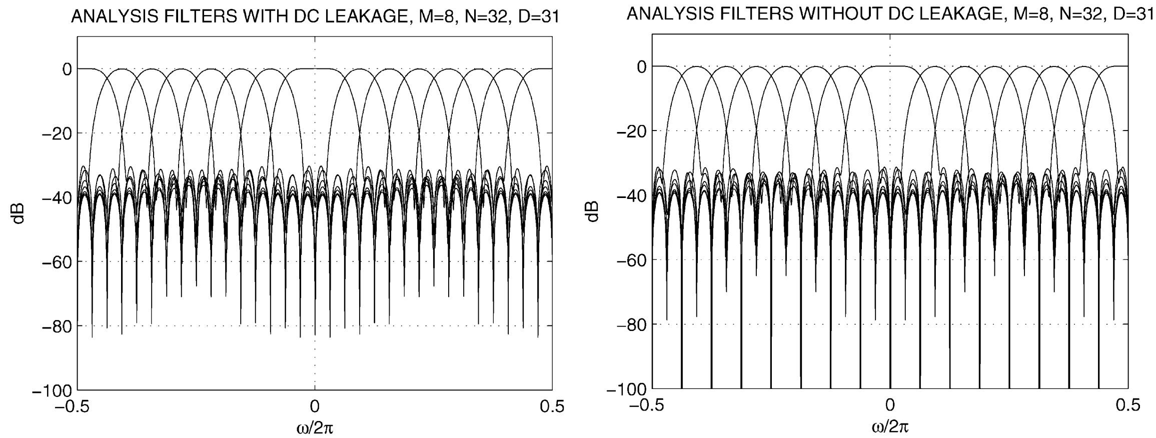

Fig. 5. Prototype filters with DC leakage (left) and without DC leakage (right); $M = 8$, $N = 32$, $D = 31$.

Table 1
Factorization of prototype into zero-delay, maximum-delay, and initialization matrices for prototypes with and without DC leakage

	Both		Both		DC leakage				No DC leakage			
ℓ	$\delta_{\ell,1}$	$d_{\ell,1}$	$\beta_{\ell,2}$	$b_{\ell,2}$	$g_{\ell,0}$	$g_{\ell,1}$	$g_{\ell,2}$	$g_{\ell,3}$	$g_{\ell,0}$	$g_{\ell,1}$	$g_{\ell,2}$	$g_{\ell,3}$
0	1	− 0.2499	1	0.2352	− 0.2421	0.0881	0.0830	0.2279	− 0.2427	0.0867	0.0816	0.2284
1	1	− 0.1601	1	0.1561	− 0.2261	0.1138	0.1110	0.2205	− 0.2276	0.1109	0.1081	0.2219
2	1	− 0.0882	1	0.0875	− 0.2084	0.1398	0.1387	0.2068	− 0.2101	0.1373	0.1362	0.2085
3	1	− 0.0280	1	0.0280	− 0.1880	0.1649	0.1648	0.1879	− 0.1888	0.1640	0.1639	0.1887

7.2. Low-delay prototype filters

We now give an example for the design of low-delay prototype filters. Again, we start from a PR linear-phase prototype filter ($M = 8$, $N = 16$, $D = 15$) designed with the QCLS algorithm and design prototype filters of greater length, resulting in the same system delay by adding successively zero-delay matrices to the factorization of our starting point prototype and using non-linear optimization for the calculation of the coefficients. The frequency responses of the resulting filters of lengths $N = 32$ and 48 as well as the starting point with $N = 16$ are depicted in Fig. 6. Increasing the filter length significantly improves the stopband attenuation while keeping the system delay constant.

8. Conclusions

In this paper, we have connected two different approaches for the design of biorthogonal cosine-modulated filter banks with perfect reconstruction. Based on the PR constraints we have shown how the polyphase filters can be realized using zero-delay and maximum-delay matrices. Especially, we have shown that all PR prototypes are covered by the new approach, which was not the case for the method in [12]. The proposed implementation structure has the advantage that it automatically guarantees PR even after coefficient quantization, so that it is suitable for VLSI designs. Furthermore, the implementation cost is nearly halved, when compared to a direct realization of the polyphase filters. We can design different

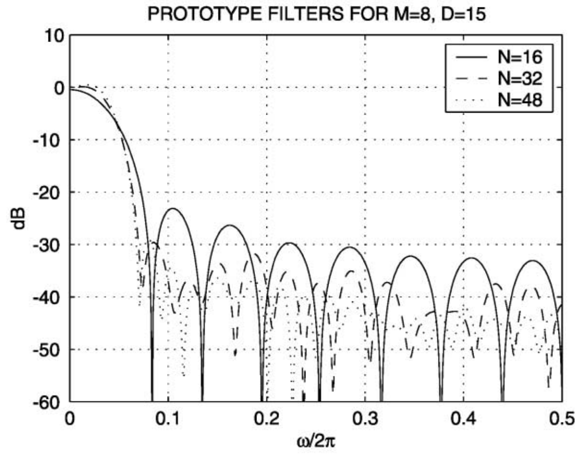

Fig. 6. Low-delay prototype filters of lengths $N = 16, 32, 48$ for a biorthogonal cosine-modulated filter bank with system delay $D = 15$ in all cases.

prototype filters for the analysis and synthesis bank, and we can also include restrictions in order to obtain one common prototype. Using a modified set of constraints for the first matrix, we can also design and implement biorthogonal cosine-modulated filter banks without DC leakage in an efficient way.

Appendix A

A.1. Calculation of maximum-delay matrices

We here show that (34)–(36) hold true for every set of PR prototype filters satisfying (10) and (11). Substituting (20) and (21) into (32) and (33), respectively, we obtain

$$\boldsymbol{G}_{\mathrm{sh}}(z) = \begin{bmatrix} z^{\delta_j - 1}(-1)^{s-1} G_{\ell+M}(-z^2) & z^{\delta_j - 1} G_{d-\ell}(-z^2) \\ zG_\ell(-z^2) - d_j z^{\delta_j}(-1)^{s-1} G_{\ell+M}(-z^2) & (-1)^s z G_{d-\ell-M}(-z^2) - d_j z^{\delta_j} G_{d-\ell}(-z^2) \end{bmatrix}, \tag{A.1}$$

$$\boldsymbol{K}_{\mathrm{sh}}(z) = \begin{bmatrix} d_j z^{\delta_j} K_{d-\ell}(-z^2) + (-1)^{s-1} z K_{d-\ell-M}(-z^2) & z^{\delta_j - 1} K_{d-\ell}(-z^2) \\ d_j z^{\delta_j}(-1)^s K_{\ell+M}(-z^2) + zK_\ell(-z^2) & z^{\delta_j - 1}(-1)^s K_{\ell+M}(-z^2) \end{bmatrix}. \tag{A.2}$$

Eqs. (34)–(36) can be rewritten as

$$g_{\ell+M}(n) = g_{d-\ell}(n) = k_{d-\ell}(n) = k_{\ell+M}(n) = 0 \quad \forall n \quad \text{with } n \geqslant 0 \text{ and } n \leqslant (\delta_j - 3)/2, \tag{A.3}$$

$$g_{\ell+M}((\delta_j - 1)/2) \neq 0, \quad g_{d-\ell}((\delta_j - 1)/2) \neq 0, \quad k_{d-\ell}((\delta_j - 1)/2) \neq 0, \quad k_{\ell+M}k((\delta_j - 1)/2) \neq 0, \tag{A.4}$$

$$\begin{aligned} d_j &= \frac{g_\ell(0)}{(-1)^{s-1} g_{\ell+M}((\delta_j - 1)/2)} = \frac{(-1)^s g_{d-\ell-M}(0)}{g_{d-\ell}((\delta_j - 1)/2)} \\ &= \frac{k_\ell(0)}{(-1)^{s-1} k_{\ell+M}((\delta_j - 1)/2)} = \frac{(-1)^s k_{d-\ell-M}(0)}{k_{d-\ell}((\delta_j - 1)/2)}. \end{aligned} \tag{A.5}$$

Eqs. (A.3) and (A.4) can be seen as equations to determine a value of δ_j that ensures that no division by zero occurs in (A.5). The structures of (A.1) and (A.2) show that δ_j needs to be odd. The value of δ_j is to be chosen to shorten the delay as much as possible, and for contiguous prototypes $\delta_j = 1$ provides the solution. If a prototype is non-contiguous, one needs to evaluate (A.3) and (A.4) in order to find an appropriate δ_j. It remains to show that (A.5) is satisfied by all PR prototypes. For this we look at the time-domain formulation of (11):

$$g_\ell(m)*k_{d-\ell}(m) + g_{M+\ell}(m)*k_{d-\ell-M}(m) = \frac{1}{2M}\delta(m-s) \quad 0 \leqslant \ell \leqslant d-M,\ \ell \neq \frac{d-M}{2}. \tag{A.6}$$

It can be verified that arbitrary parts of the equalities in (A.5) satisfy (A.6) with $m = (\delta_j - 1)/2 \neq s$. For the most common case of contiguous prototype filters where $\delta_j = 1$ the result is

$$g_\ell(0)k_{d-\ell}(0) + g_{\ell+M}(0)k_{d-\ell-M}(0) = 0, \quad k_\ell(0)g_{d-\ell}(0) + k_{\ell+M}(0)g_{d-\ell-M}(0) = 0, \tag{A.7}$$

$$g_\ell(0)k_{\ell+M}(0) - g_{\ell+M}(0)k_\ell(0) = 0, \quad g_{d-\ell}(0)k_{d-\ell-M}(0) - g_{d-\ell-M}(0)k_{d-\ell}(0) = 0. \tag{A.8}$$

Note that no other constraints than those in (A.6) are imposed by (A.5). Thus, also (36) holds true, and it is clear that the factorization can be performed for every PR prototype satisfying (10) and (11) which were orginally derived in [1].

A.2. Calculation of zero-delay matrices

Here we show that (46)–(47) hold true for every set of PR prototype filters satisfying (10) and (11). Substituting (20) and (21) into (44) and (45), respectively, we obtain

$$\boldsymbol{G}_{\mathrm{sh}}(z) = \begin{bmatrix} -b_i z^{-\beta_i} G_\ell(-z^2) + (-1)^{s-1} z^{-1} G_{\ell+M}(-z^2) & -b_i z^{-\beta_i} G_{d-\ell-M}(-z^2) + z^{-1} G_{d-\ell}(-z^2) \\ G_\ell(-z^2) & (-1)^s G_{d-\ell-M}(-z^2) \end{bmatrix}, \tag{A.9}$$

$$\boldsymbol{K}_{\mathrm{sh}}(z) = \begin{bmatrix} (-1)^{s-1} K_{d-\ell-M}(-z^2) & z^{-1} K_{d-\ell}(-z^2) + b_i z^{-\beta_i} (-1)^{s-1} K_{d-\ell-M}(-z^2) \\ K_\ell(-z^2) & (-1)^s z^{-1} K_{\ell+M}(-z^2) + b_i z^{-\beta_i} K_\ell(-z^2) \end{bmatrix}. \tag{A.10}$$

The delay β_i plays the same role as δ_j in the above derivation, and from the form of (A.9) and (A.10) it is evident that β_i will always have an odd value. Denoting by N_ℓ, $N_{\ell+M}$, $N_{d-\ell}$, and $N_{d-\ell-M}$ the lengths of the polyphase filters $G_\ell(z)$, $G_{\ell+M}(z)$, $G_{d-\ell}(z)$, and $G_{d-\ell-M}(z)$, Eq. (47) can be written as

$$b_i = \frac{(-1)^{s-1} g_{\ell+M}(N_\ell - 1 + (\beta_i - 1)/2)}{g_\ell(N_\ell - 1)} = \frac{g_{d-\ell}(N_{d-\ell-M} - 1 + (\beta_i - 1)/2)}{(-1)^s g_{d-\ell-M}(N_{d-\ell-M} - 1)} \tag{A.11}$$

$$= \frac{(-1)^{s-1} k_{\ell+M}(N_\ell - 1 + (\beta_i - 1)/2)}{k_\ell(N_\ell - 1)} = \frac{k_{d-\ell}(N_{d-\ell-M} - 1 + (\beta_i - 1)/2)}{(-1)^s k_{d-\ell-M}(N_{d-\ell-M} - 1)}. \tag{A.12}$$

Using the fact that the polyphase filters belong to PR prototypes, one can derive from (A.6) that the following relationship for lengths of the polyphase filter holds true:

$$N_\ell + N_{d-\ell} = N_{\ell+M} + N_{d-\ell-M} \Leftrightarrow N_{\ell+M} - N_\ell = N_{d-\ell} - N_{d-\ell-M}. \tag{A.13}$$

A relationship equivalent to (46), expressed by the polyphase components lengths, writes

$$\beta_i = 2(N_{\ell+M} - N_\ell) + 1 = 2(N_{d-\ell} - N_{d-\ell-M}) + 1. \tag{A.14}$$

The constraints imposed by (A.11) on the polyphase filters are no other than those imposed by the PR constraint (A.6) and are thus automatically satisfied for every PR prototype.

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